

## **Inductive constructions for Lie bialgebras and Hopf algebras**

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# Inductive constructions for Lie bialgebras and Hopf algebras

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# Abstract

In recent years, two generalisations of the theory of Lie algebras have become prominent, namely the “semi-classical” theory of Lie bialgebras and the “quantum” theory of Hopf algebras, including the quantized enveloping algebras. I develop an inductive approach to the study of these objects.

An important tool is a construction called double-bosonisation defined by Majid for both Lie bialgebras and Hopf algebras, inspired by the triangular decomposition of a Lie algebra into positive and negative roots and a Cartan subalgebra. We describe two specific applications. The first uses double-bosonisation to add positive and negative roots and considers the relationship between two algebras when there is an inclusion of the associated Dynkin diagrams. In this setting, which we call *Lie induction*, double-bosonisation realises the addition of nodes to Dynkin diagrams. We use our methods to obtain necessary conditions for such an induction to be simple, using representation theory, providing a different perspective on the classification of simple Lie algebras.

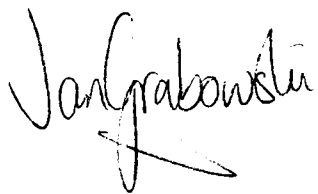
We consider the corresponding scheme for quantized enveloping algebras, based on inclusions of the associated root data. We call this *quantum Lie induction*. We prove that we have a double-bosonisation associated to these inclusions and investigate the structure of the resulting objects, which are Hopf algebras in braided categories, that is, covariant Hopf algebras.

The second application generalises one of the most important constructions in this field, namely the Drinfel’d double of a Lie bialgebra, which has dimension twice that of the underlying algebra. Our construction, the *triple*, has dimension three times that of the input algebra. Our main result is that when the input algebra is factorisable, this is isomorphic to the triple direct sum as an algebra and a twisting as a coalgebra. We also indicate a number of ways in which the triple is related to the double.

# Statement of Originality

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Signed:

A handwritten signature in black ink, appearing to read "Jan Grabowski". The signature is written in a cursive style with a large, sweeping initial 'J' and a long, horizontal flourish at the end.

Date:

A handwritten date in black ink, reading "13th April 2006". The text is written in a cursive style with a large, sweeping initial 'A'.

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*For Jennifer*

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*“For the LORD gives wisdom, and from his mouth come knowledge  
and understanding”*

Proverbs 2:6

# Chapter 1

## Introduction

The study of Lie algebras is long-established and widely utilized but in recent years, two generalisations of the theory have become prominent and almost as ubiquitous. These are the “semi-classical” theory of Lie bialgebras and the “quantum” theory of quantized enveloping algebras. In this scheme, the original theory is called “classical”. This thesis is concerned with the an inductive approach to the study of these objects.

We make use of a tool developed in the quantum theory for the classical, via the semi-classical, namely a construction called double-bosonisation defined by Majid. Double-bosonisation is defined for both Lie bialgebras and Hopf algebras and was inspired by the triangular decomposition of a Lie algebra into positive and negative roots and a Cartan subalgebra. In the semi-classical setting, we use double-bosonisation to add positive and negative roots and consider the relationship between two Lie (bi-)algebras when there is an inclusion of the associated Dynkin diagrams. We call this *Lie induction* and here double-bosonisation realises the addition of nodes to Dynkin diagrams. Our motivation here is to study when such an induction is simple, using representation theory, to provide a different perspective on the classification of simple Lie algebras.

Next, we turn our attention to the quantum setting, to examine this inductive approach for quantized enveloping algebras. Using Lusztig’s abstract approach of considering root data, we may again consider the relationship between two quantized enveloping algebras when there is an inclusion of the associated root data, and hence the Dynkin diagrams. Correspondingly, we call this *quantum Lie induction*. We show that this may be done in a very general setting and study in detail the associated structures. The aim is to gain further insight into the structure of quantized enveloping algebras and suggest

how one might prove various of their properties by an inductive method.

We then consider an inductive construction for Lie bialgebras which is very different from Lie induction, although inspired by it and with double-bosonisation also underpinning it. We generalise one of the most important constructions in the field, namely the Drinfel'd double of a Lie bialgebra, which has dimension twice that of the underlying algebra. Our construction, the *triple*, has dimension three times that of the input algebra. The definition of the triple is via a natural double-bosonisation and our main result is that when the input algebra is factorisable, this is isomorphic to the triple direct sum as an algebra. The coalgebra structure is not trivial but is a natural twisting. We also indicate a number of ways in which the triple is related to the double.

To enable us to expand on these descriptions, we briefly introduce the structures and constructions we need. The full details may be found in Chapter 2. We also discuss our motivations for this work.

We recall that a Lie algebra  $\mathfrak{g}$  over a field  $k$  is a  $k$ -vector space equipped with a map  $[\ , \ ] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ , called the bracket, satisfying  $[x, x] = 0$  for all  $x \in \mathfrak{g}$  and (the Jacobi identity)

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{for all } x, y, z \in \mathfrak{g}.$$

A Lie bialgebra is a Lie algebra with a cobracket structure  $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  satisfying axioms precisely dual to those for a Lie algebra, with an appropriate compatibility condition. The definition, due to Drinfel'd ([Dri83]), is comparable with that of a Hopf algebra  $H$ , where we have a multiplication map  $m : H \otimes H \rightarrow H$  and a compatible comultiplication  $\Delta : H \rightarrow H \otimes H$ . The comultiplication defines an algebra structure on the dual  $H^*$ , so we think of a Hopf algebra as being self-dual in this sense. The definition of a Lie bialgebra is the semi-classical version or *infinitesimalisation* of this. Moreover, Lie bialgebras exponentiate geometrically to Poisson–Lie groups with Poisson bracket linearizing to  $\delta$  and have been of considerable interest to Poisson and symplectic geometers.

An important class of Lie bialgebras is that of quasitriangular Lie bialgebras. Here, the cobracket  $\delta$  is of a specific form, namely, the coboundary of an element  $r \in \mathfrak{g} \otimes \mathfrak{g}$  satisfying two conditions, one of which is the well-known classical Yang–Baxter equa-

tion. We will mostly be concerned with the Lie algebra structures when considering Lie induction. (However, a quasitriangular bialgebra structure is absolutely essential to the construction we use.) We will usually take the canonical Drinfel'd–Sklyanin cobracket, leading to the Drinfel'd–Sklyanin Poisson bracket on the associated Lie group, which does give a quasitriangular Lie bialgebra.

We also need a generalisation of the notion of a Lie bialgebra, that of a braided-Lie bialgebra. Given a quasitriangular Lie bialgebra  $\mathfrak{g}$ , this is a  $\mathfrak{g}$ -covariant bialgebra in the category of  $\mathfrak{g}$ -modules, where the cobracket  $\underline{\delta}$  has non-zero coboundary. In fact  $d\underline{\delta} = \psi$ , where  $\psi$  is a canonical braiding operator. This definition has been given by Majid ([Maj00]) as the infinitesimal version of Hopf algebras in braided categories, which we describe below.

Our principal tool here is the double-bosonisation construction of Majid for Lie bialgebras ([Maj00]). Assuming all our objects to be finite-dimensional, we take as input to the construction a quasitriangular Lie bialgebra  $\mathfrak{g}_0$  and a braided-Lie bialgebra  $\mathfrak{b}$ . One then obtains a new quasitriangular Lie bialgebra  $\mathfrak{g} = \mathfrak{b} \bowtie \mathfrak{g}_0 \bowtie \mathfrak{b}^{*\text{op}}$ . (In general, one takes two dually paired braided-Lie bialgebras but we always take the usual dual in the finite-dimensional case.) This may be defined over a field of characteristic not two but we will now restrict to working over  $\mathbb{C}$ .

We ask two questions:

- i) Given a Lie bialgebra  $\mathfrak{g}$  and a sub-bialgebra  $\mathfrak{g}_0$ ,
  - (a) can we find a braided-Lie bialgebra  $\mathfrak{b}$  such that we may reconstruct  $\mathfrak{g}$  by double-bosonisation and
  - (b) if so, what is its structure?
- ii) Given a Lie bialgebra  $\mathfrak{g}_0$ , for which choices of braided-Lie bialgebra  $\mathfrak{b}$  do we obtain something “interesting” on taking the double-bosonisation?

For the first, we concentrate on the semisimple case with  $\text{rank } \mathfrak{g}_0 = \text{rank } \mathfrak{g} - 1$ . We refer to this as the *corank one* case. In this situation, as shown by Majid, we have a positive answer to the first part. Of course, we may repeat the process to answer the question

for higher coranks. The answer to the second part is the content of Section 3.1 and the Appendix of this thesis.

An example of a negative answer to the first part is the corank zero case, that is  $\text{rank } \mathfrak{g}_0 = \text{rank } \mathfrak{g}$ . In fact, the cases are quite different in character. Although we may attack both representation-theoretically by restricting the adjoint representation of the larger algebra to the chosen subalgebra, when we consider the brackets on the modules we obtain, we have very different behaviour. In the corank one case we have  $\mathbb{Z}$ -gradings, but in the corank zero case we see  $\mathbb{Z}/(2)$ - and  $\mathbb{Z}/(3)$ -gradings. We cannot use double-bosonisation to reconstruct the larger algebra in the corank zero case, as it cannot reproduce these gradings by finite groups. We note that the construction attributed to Freudenthal in Chapter 22 of [FH91] deals with the  $\mathbb{Z}/(3)$ -graded case purely on the level of Lie algebras.

For “interesting” in the second question above, we take the finite-dimensional simple complex Lie algebras, with the Drinfel’d–Sklyanin Lie cobracket. We consider in Section 3.2 the obvious induction scheme coming from Dynkin diagrams, namely, since we know that simple Lie algebras have connected Dynkin diagrams, adding a new node to a diagram in as many ways as is possible. Clearly not all choices are allowed and we give some conditions and results, purely algebraic in nature, which control this. We do not have a complete list of such conditions, so we cannot re-prove the classification of the simple complex Lie algebras, but we do have enough necessary conditions to exclude some possibilities immediately. In particular, we discuss the induction from the maximal rank exceptional Lie algebras.

There has long been interest in the exceptional series of simple complex Lie algebras, for a variety of reasons in both mathematics and physics. One goal has been to find a unified construction for these algebras. Double-bosonisation goes beyond this, giving a unified construction of *all* simple complex Lie algebras. As we show here, it does so in a way that is completely compatible with the natural inclusions of Dynkin diagrams. Fully understanding Lie induction across the range of cases it encompasses is a programme that goes beyond the present work. Here we set out some underlying theory and demonstrate

the method for the exceptional series.

This, on its own, is a sufficient motivation to study double-bosonisation but there are others. Firstly, it confirms the necessity of considering Lie bialgebras—not just Lie algebras—and their braided versions. Also, through our calculations here, we have identified a number of new examples of braided-Lie bialgebra structures.

A second important goal has been to understand why we obtain only the infinite series and exceptionals we know in the classification. The original proof of the classification is essentially a combinatorial analysis of a set of geometric objects, namely root systems. Lie induction via double-bosonisation deals directly with the algebras and their representation theory. In doing so, it emphasises the strong relationship between the structure theory and the representation theory, which has become a theme in algebra. Using Lie induction, we are able to formulate in a new way the question of the obstruction to finding, for example, a finite-dimensional Lie algebra of type  $E_9$ .

Our study of Lie induction in Chapter 3 falls into two complementary but intimately connected halves, Section 3.1 discussing deletion and Section 3.2 on induction. We know from [Maj00] that to each choice of simple root in a semisimple Lie algebra  $\mathfrak{g}$  we have associated a braided-Lie bialgebra  $\mathfrak{b}$  such that we can recover  $\mathfrak{g}$  by double-bosonisation from  $\mathfrak{g}_0$ , where  $\mathfrak{g}_0$  is a Lie subalgebra with Dynkin diagram that of  $\mathfrak{g}$  with the node corresponding to the chosen simple root deleted. We refer to calculating these  $\mathfrak{b}$  as *deletion*.

Then the aim of Section 3.1 is to further analyse the structure of this  $\mathfrak{b}$  and provide some tools for calculating  $\mathfrak{b}$  explicitly. Critical to this are Lemma 3.1.1, where we observe that associated to each simple root is a decomposition of  $\mathfrak{g}$  which defines a  $\mathbb{Z}$ -grading, and Proposition 3.1.2, where we cite a result of Azad, Barry and Seitz ([ABS90]) which tells us that the homogeneous components of this grading are irreducible modules for the zeroth part (except the zeroth part itself, which is not irreducible).

We have calculated the braided-Lie bialgebra structures associated to simple Lie algebras  $\mathfrak{g}$  in the case where we delete a simple root corresponding to an extremal node in the Dynkin diagram, so that the subalgebra  $\mathfrak{g}_0$  is simple. That is, for deletion to a



simple Levi subalgebra. We record the full results of our calculations in an Appendix and provide a summary table (Table 3.3) of the modules we find for each deletion in Section 3.1.3. We also consider the rôle of the graph automorphisms of the Dynkin diagrams, in Section 3.1.2.

In Section 3.2, we take the opposite view and ask for necessary conditions on braided-Lie bialgebras  $\mathfrak{b}$  in general to obtain finite-dimensional simple Lie bialgebras via double-bosonisation. Some properties are immediate from our previous analysis of deletion, for example, that  $\mathfrak{b}$  must be  $(\mathbb{Z})$ -graded. We list these in Section 3.2.1.

From this list, we have two properties of particular importance, namely that the irreducible graded components of  $\mathfrak{b}$  should have all their weight spaces one-dimensional (property (3) on page 58) and should have at most two dominant weights (property (4')). We call a module with these two properties a *defining* module. In Section 3.2.2, we record a result communicated to us by Y. Bazlov which classifies the defining modules for the simple complex Lie algebras.

This forms the basis of our analysis in Section 3.2.3 of some of the obstructions to the existence of simple exceptional Lie algebras other than those already known. Specifically, we see that there exist no non-trivial defining modules for  $E_8$ , and so we cannot produce a finite-dimensional algebra of type  $E_9$ . Analysis of possible inductions from  $A_8$  and  $D_8$  only reinforce this. Although we obtain some potential candidates from each, we have none that are consistent. We make similar analyses for  $F_5$  and  $G_3$ .

However, this work has implications outside of Lie algebra theory. In recent years, the quantized enveloping (Hopf) algebra  $U_q(\mathfrak{g})$  associated to a Lie algebra  $\mathfrak{g}$  has become one of the most significant objects of study among algebraists, geometers and physicists alike. These objects are still not fully understood, although much progress has of course been made.

The quantized enveloping algebras are defined relative to an abstraction of the notion of a root system for Lie algebras, called a root datum. They are quantizations of the universal enveloping algebra of a Lie algebra and are Hopf algebras. We note that they are not quite quasitriangular, due to the fact that they are infinite-dimensional, but

there is an alternative formulation for the axioms of a quasitriangular structure which resolves this.

There is a strong relationship between what we refer to as the semi-classical theory, that of Lie bialgebras, and the quantum, in the form of  $U_q(\mathfrak{g})$ . Most notably from our perspective, double-bosonisation has also been defined for Hopf algebras, in work of Majid ([Maj99]) and, independently and in a special case, Sommerhäuser ([Som96]). So, we might hope that the methods we developed for Lie bialgebras should carry over to the quantum setting, providing some new insights into the structure of  $U_q(\mathfrak{g})$ . Indeed this is the case.

In place of braided-Lie bialgebras, we have braided groups: these are Hopf algebras in braided categories. A braided category is a generalisation of a symmetric monoidal category, where we have a natural isomorphism  $\Psi$  defined by isomorphisms

$$\Psi_{A,B} : A \otimes B \rightarrow B \otimes A \quad \text{for all objects } A, B$$

but the  $\Psi_{A,B}$  are not (necessarily) the tensor product flip map,  $\tau : a \otimes b \mapsto b \otimes a$ . However,  $\Psi$  does satisfy two hexagon identities and the Yang–Baxter equation holds, in the form

$$\Psi_{A,B} \circ \Psi_{A,C} \circ \Psi_{B,C} = \Psi_{B,C} \circ \Psi_{A,C} \circ \Psi_{A,B}$$

for all objects  $A, B, C$ . The prototype example is that of the module category of a quasitriangular Hopf algebra.

We may consider algebraic structures on objects in such categories, where the maps such as the product or coproduct are morphisms in the category between appropriate objects. For a braided group, we have all the maps defining a Hopf algebra structure as morphisms in the category, for example  $\underline{\Delta} : B \rightarrow B \otimes B$ . Here  $\otimes$  denotes the braided tensor product algebra structure:  $(a \otimes b)(c \otimes d) = a\Psi(b \otimes c)d$ . As yet, relatively few examples are known but it is clear that an analysis such as we have carried out for Lie bialgebras will provide a large class of examples, which is an additional motivation for our work.

In Chapter 4, we study quantum Lie induction. We start with the definition of sub-root data (Section 4.1), that is, a pair of suitably related root data, denoted  $\mathfrak{J} \subseteq_l \mathfrak{T}$ .

This notion generalises that of the inclusions of Dynkin diagrams we have discussed above. We define an induced map on the weight lattice, which encodes the restriction of representations from the larger algebra to the smaller.

As in the semi-classical case,  $\mathbb{Z}$ - and  $\mathbb{N}$ -gradings are important as a tool for simplifying the analysis of the structures involved, as we can consider the homogeneous components one at a time. We consider  $\mathbb{N}$ -gradings on Hopf algebras (Section 4.2) and show that we always obtain a split Hopf algebra projection  $H \xrightarrow[\iota]{\pi} H_0$  onto the zeroth component. The Radford–Majid theorem then tells us that we have a Hopf algebra  $B$  in the braided category of  $D(H_0)$ -modules ( $D$  the Drinfel’d quantum double) such that  $B \bowtie H_0 \cong H$ , a (single) bosonisation. The bosonisation is a certain semi-direct product and coproduct Hopf algebra. Some analysis is possible even at this level of generality: we prove that  $B$  inherits a  $\mathbb{N}$ -grading from  $H$  (Theorem 4.2.6).

The reason for considering such a setup is that the analogue of the negative Borel subalgebra in the quantum setting, denoted  $U_q^\leq(\mathfrak{T})$ , is  $\mathbb{N}$ -graded and hence these results apply (Section 4.3) and we obtain  $B = B(\mathfrak{T}, \mathfrak{J}, \iota)$ , a braided group. In Theorem 4.3.4, we show that the zeroth component of this grading is a central extension of  $U_q^\leq(\mathfrak{J})$ , the quantum negative Borel subalgebra of the smaller algebra. Then we may use the idea of Drinfel’d of constructing  $U_q(\mathfrak{T})$  as a quotient of the double to prove that  $U_q(\mathfrak{T})$  is a double-bosonisation of  $U_q(\mathfrak{J})$  (Theorem 4.4.2).

In Section 4.5, we analyse the algebra, module and coalgebra structures of  $B$ . We give a set of generators for  $B$  (Theorem 4.5.1), show that its first homogeneous component  $B_1$  is a direct sum of quotients of Weyl modules (Proposition 4.5.8) and show that  $B$  is in fact integrable (Theorem 4.5.9). We cannot say so much about the coalgebra structure but we give some initial observations.

In Chapter 5 we return to the semi-classical setting and describe a second type of inductive construction. Probably the most important of all Lie bialgebra constructions is the Drinfel’d double that associates to any Lie bialgebra  $\mathfrak{g}$  a quasitriangular one  $D(\mathfrak{g}) = \mathfrak{g} \bowtie \mathfrak{g}^{\text{op}}$ . Here it is presented as a double cross sum of  $\mathfrak{g}$  and its dual acting on each other. When  $\mathfrak{g}$  is quasitriangular one also knows that  $D(\mathfrak{g})$  is isomorphic to a

‘bosonisation’  $\mathfrak{g} \bowtie \underline{\mathfrak{g}}^{*\text{op}}$  as well as a cocycle twist  $(\mathfrak{g} \oplus \mathfrak{g})_\chi$  when  $\mathfrak{g}$  is factorisable (when the symmetric part of  $r$  is non-degenerate). Here  $\underline{\mathfrak{g}}$  denotes the braided-Lie bialgebra associated to  $\mathfrak{g}$  by transmutation ([Maj00]), a canonical braided-Lie bialgebra structure on the adjoint representation of  $\mathfrak{g}$ .

There are, however, some defects of the Drinfel’d double and these are solved in a canonical ‘triple’ construction  $T(\mathfrak{g})$  which we formulate and study in Chapter 5. First of all, the Drinfel’d double does not respect the Cartan decomposition of Lie groups and Lie algebras into positive roots, Cartan subalgebra and negative roots, so cannot be used to construct them directly. (Drinfel’d here, and in the quantum version to construct quantum groups, uses the double of the Borel subalgebra and then identifies the two Cartan subalgebras via a quotient, an idea we use in our study of quantum Lie induction.) Related to this, the double is a special case of constructions which do not in general yield quasitriangular Lie bialgebras. By contrast our triple

$$T(\mathfrak{g}) \stackrel{\text{def}}{=} \underline{\mathfrak{g}} \bowtie \mathfrak{g} \bowtie \underline{\mathfrak{g}}^{*\text{op}}$$

is a canonical example of a triple product construction, via the double-bosonisation  $\mathfrak{b} \bowtie \mathfrak{g} \bowtie \mathfrak{b}^*$  ([Maj00]) which is always quasitriangular and which respects the Cartan decomposition of simple Lie algebras (and of quantum groups in the quantum case). The special case in which  $\mathfrak{b} = \underline{\mathfrak{g}}$  and the actions are (co)adjoint is canonical and provides our natural extension of the double. After formulating  $T(\mathfrak{g})$  we examine its structure and prove that it is indeed an extension of the Drinfel’d double. Our principal result (Theorem 5.2.5) is the isomorphism of  $T(\mathfrak{g})$  with  $(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g})_\chi$ , the twisting of the direct sum bialgebra by a cocycle. This is precisely analogous to that for the double mentioned above. Another result is that  $T(\mathfrak{g})$  is isomorphic to  $\underline{\mathfrak{g}} \bowtie D(\mathfrak{g})$  (Corollary 5.3.2), i.e. a double cross sum as a Lie algebra and semidirect as a Lie coalgebra.

Apart from its internal structure, some secondary motivations are as follows. In our study of Lie induction, we asked why there is no  $E_9$  arising from  $\mathfrak{g} = E_8$ , for which we would need to consider braided-Lie bialgebras in the category of modules of  $E_8$ . Since the smallest non-trivial representation of  $E_8$  is the adjoint one, we were naturally led to the quasitriangular Lie bialgebra  $\underline{E_8} \bowtie E_8 \bowtie \underline{E_8} = T(E_8)$  as the smallest candidate

for  $E_9$  from this point of view. Our Theorem 5.2.5 tells us that this is not simple as a Lie algebra, as expected, but note that it is non-trivial and not a direct sum as a Lie bialgebra. (Also, the adjoint representation fails the tests for valid candidates for simple inductions in Section 3.2.1.)

Secondly, Lie bialgebras (and the Drinfel'd double in particular) play a central role in mathematical physics—in the theory of integrable systems and more recently in string theory. For example, both the Lie bialgebra  $D(\mathfrak{g})$  and its corresponding Poisson-Lie group  $D(G)$  occur when one considers Poisson-Lie  $T$ -duality ([Kli96]) in  $\sigma$ -models. The algebraic structure behind this has been analysed in [BM01], making central use of the cotwist theorem for the Drinfel'd double mentioned above. Beggs and Majid also generalise  $T$ -duality to more general double cross sums. It seems likely then that our extension  $T(\mathfrak{g})$ , which has both the cotwist property and can be represented as a double cross sum, should be the natural basis for an extension of Poisson-Lie  $T$ -duality, possibly in the context of higher order dualities based on a ‘triple product’ factorisation rather than a binary factorisation. Likewise for other models in mathematical physics where the Drinfel'd double is used.

Also, the results about the Drinfel'd double infinitesimalise results about the Drinfel'd quantum double in terms of braided groups and Hopf algebra twisting respectively. We should then define the *quantum triple*  $T(H)$  of a Hopf algebra  $H$  in the obvious way and the information we have obtained about the (semi-classical) triple should give us insight into the structure of the quantum triple. We discuss the quantum triple a little further in Section 6.3.

The structure of Chapter 5 is as follows. Section 5.1 defines and studies the triple  $T(\mathfrak{g})$  and contains our first simplifications of its algebraic structure in the factorisable case. Section 5.2 contains our main results that the Lie algebra structure is isomorphic to the direct sum  $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$  (Theorem 5.2.3) and a cocycle twisting as a Lie bialgebra (Theorem 5.2.5). Note that these results do not imply that the triple  $T(\mathfrak{g})$  is trivial any more than for  $D(\mathfrak{g})$ . Next, in Section 5.3, we give several theorems relating the triple to the Drinfel'd double, as an extension, in various ways.

All these general results are over any field of characteristic not 2. In Section 5.4, we work over  $\mathbb{C}$  and examine the half-real forms of the triple. A half-real form is a choice of basis of a complex bialgebra with real Lie algebra structure constants and imaginary coalgebra structure constants. We prove that if  $(\mathfrak{u}, r)$  is a half-real form of  $(\mathfrak{g}, r)$  then  $T(\mathfrak{u})$ , defined to be the appropriate double-bosonisation, is a half-real form of  $T(\mathfrak{g})$ , under a natural reality assumption on  $r$ , and that  $T(\mathfrak{u})$  has its quasitriangular structure also of this real type.

Finally, for completeness, we discuss the case when our input bialgebra is triangular, the opposite extreme from factorisable. Whereas factorisability is the property that the symmetric part  $2r_+$  of the quasitriangular structure defines an isomorphism of  $\mathfrak{g}$  with  $\mathfrak{g}^*$ , triangularity is the case where this symmetric part is zero. We examine this situation in Section 5.5.

We conclude with some remarks on further extensions of Lie induction, quantum Lie induction and the triple.

It is our opinion that the results we have obtained here and the applications described above strengthen the argument that among the variety of double cross products and coproducts, the study of double-bosonisation in particular is well justified. Much further work is clearly necessary, in particular for the quantum case, but we hope this work will stimulate that.

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The material in Chapters 3 and 5 on Lie induction and the triple construction has been published as [Gra05a] and [Gra05b] respectively.

## Chapter 2

# Preliminaries

In this chapter, we recall the definitions of the structures we are interested in, together with some associated results and constructions. We divide these into three settings, using the terminology we introduced above. The first, the classical setting, is that of Lie algebras. We introduce Lusztig's abstract notions of Cartan data and root data and set some notations.

The second setting is the semi-classical, which refers to the theory of Lie bialgebras. These, as we shall see, are Lie algebras with an additional structure, dual to the bracket. We will describe these in more detail and introduce several related structures and constructions. All will be important in the sequel but of particular note is the definition of a braided-Lie bialgebra, a braided version of a Lie bialgebra.

Finally, we have the quantum setting, namely bialgebras and Hopf algebras. Like Lie bialgebras, the definition of a bialgebra has a 'self-duality'. Bialgebras are algebras with additional structures: a comultiplication and a counit. Hopf algebras have one further structure, an antipode. It is often observed that the antipode on a Hopf algebra plays the rôle of an inverse. Bialgebras are thought of as 'quantum monoids' and Hopf algebras as 'quantum groups'. Again, we particularly note the braided setting: that of Hopf algebras in braided categories, also called 'braided groups'. We also introduce quantized enveloping algebras associated to root data.

For both the semi-classical and quantum settings, we give the definitions of the bosonisation and double-bosonisation constructions of Majid. These form the basis for the inductive constructions of the title of this thesis.

Throughout, we will use the following conventions regarding the natural numbers:  $\mathbb{N} = \{0, 1, 2, \dots\}$ ,  $\mathbb{N}^* = \{1, 2, 3, \dots\}$ . That is, for us  $\mathbb{N}$  is a monoid. (Recall that a semigroup is a set with an associative binary operation and a monoid is a semigroup with an identity element for the binary operation.)

In all three settings, we will use the following notations when dealing with tensor products. We use  $\tau$  to mean the tensor product flip map, e.g.  $\tau : V \otimes W \rightarrow W \otimes V$ ,  $\tau(v \otimes w) = w \otimes v$  for all  $v \in V$ ,  $w \in W$ , on any appropriate pair of vector spaces. We adopt the Sweedler notation for elements of tensor product structures, i.e. linear combinations of elementary tensors. That is, we use upper or lower parenthesized indices to indicate the placement in the tensor product, e.g.  $\sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)} \in A \otimes A \otimes A$ . We will usually drop the summation sign, as in the Einstein convention.

## 2.1 The classical setting: Lie algebras

Let  $k$  be a field (of any characteristic). A Lie algebra  $\mathfrak{g}$  over  $k$  is a  $k$ -vector space with a bilinear map  $[\ , \ ] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  called the bracket, satisfying  $[a, a] = 0$  for all  $a \in \mathfrak{g}$  and the identity

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

for all  $a, b, c \in \mathfrak{g}$ . This is the Jacobi identity. The above imply anti-symmetry, i.e.  $[a, b] + [b, a] = 0$  for all  $a, b \in \mathfrak{g}$ .

We assume that the reader is familiar with the theory of complex semisimple Lie algebras, as can be found in [Ser87], [Hum78] or [FH91], for example. Recall that such a Lie algebra has an associated root system and that the information in the root system is sufficient to distinguish two algebras, up to isomorphism. We may therefore take the view that the information in the root system is the more fundamental and, indeed, this perspective unifies the aspects of the semi-classical and quantum theories that we are concerned with.

### 2.1.1 Root data and Serre's presentation

We follow Lusztig ([Lus93]) in defining Cartan data and root data.



**Definition 2.1.1.** A Cartan datum is a finite set  $I$  and a symmetric bilinear form  $\cdot$  on  $\mathbb{Z}[I]$  such that

i)  $i \cdot i \in 2\mathbb{N}^*$  and

ii)  $C_{ij} \stackrel{\text{def}}{=} \frac{2i \cdot j}{i \cdot i} \in \{0, -1, -2, \dots\}$  for all  $i \neq j \in I$ .

The  $C_{ij}$  form the Cartan matrix and we define  $c_i \stackrel{\text{def}}{=} \frac{i \cdot i}{2}$ . We may identify certain classes of Cartan data, as follows.

**Definition 2.1.2** ([Lus93, Section 2.1.3]). Let  $\mathfrak{T} = (I, \cdot)$  be a Cartan datum. We say that  $\mathfrak{T}$  is

- of finite type if the (symmetric) matrix  $M = (i \cdot j)_{i,j \in I}$  is positive definite (that is, has only strictly positive eigenvalues).
- irreducible if  $I \neq \emptyset$  and for any  $i \neq j$  there exists a sequence  $i = i_1, i_2, \dots, i_n = j$  in  $I$  such that  $i_p \cdot i_{p+1} < 0$  for  $p = 1, 2, \dots, n-1$ .
- of affine type if  $\mathfrak{T}$  is irreducible and  $M$  as above is positive semi-definite but not positive definite. That is,  $M$  has no negative eigenvalues and zero occurs with multiplicity at least one.

We may use Serre's presentation (cf. [Ser87, p.52]) for complex semisimple Lie algebras to work only in terms of generators and relations. Let  $\mathfrak{T} = (I, \cdot)$  be a Cartan datum and let  $l = |I|$ . Form the Cartan matrix  $C = C_{ij}$ . Let  $\mathfrak{g}$  be the complex Lie algebra defined by the  $3l$  generators  $X_i^-, H_i, X_i^+$  for  $1 \leq i \leq l$  and the relations

$$\begin{aligned}
 [H_i, H_j] &= 0 & \text{for all } 1 \leq i, j \leq l, \\
 [H_i, X_j^+] &= C_{ij} X_j^+ & \text{for all } 1 \leq i, j \leq l, \\
 [H_i, X_j^-] &= -C_{ij} X_j^- & \text{for all } 1 \leq i, j \leq l, \\
 [X_i^+, X_j^-] &= \delta_{ij} H_i & \text{for all } 1 \leq i, j \leq l, \\
 [X_i^+, [X_i^+, \dots [X_i^+, X_j^+]] \dots] &= 0 & \text{for } i \neq j, 1 - C_{ij} \text{ brackets,} \\
 [X_i^-, [X_i^-, \dots [X_i^-, X_j^-]] \dots] &= 0 & \text{for } i \neq j, 1 - C_{ij} \text{ brackets.}
 \end{aligned}$$

If  $\mathfrak{T}$  is of finite type,  $\mathfrak{g}$  is a finite-dimensional complex semisimple Lie algebra.

However, working with general fields, one would like to be able to carry as much algebraic group information as possible. The concept of a root datum allows us to do this and, for example, distinguish the quantized enveloping algebras  $U_q(\mathfrak{sl}_2)$  and  $U_q(\mathfrak{psl}_2)$  (recall that  $SL(2, \mathbb{C})$  is the double cover of  $PSL(2, \mathbb{C})$ ).

**Definition 2.1.3.** A root datum  $\mathfrak{T}$  is a Cartan datum  $(I, \cdot)$  together with

- i) two finitely generated free Abelian groups  $Y, X$  with a non-degenerate pairing  $\langle \cdot, \cdot \rangle : Y \times X \rightarrow \mathbb{Z}$  and
- ii) inclusions  $Y \xleftarrow{i_1} I \xrightarrow{i_2} X$  such that  $\langle i_1(i), i_2(j) \rangle = C_{ij}$ .

Here,  $X$  plays the rôle of the character lattice, into which the root lattice is embedded;  $Y$  corresponds to the cocharacter lattice.

### 2.1.2 Lie algebra conventions

Our notations will typically follow those of [Ser87] and [Hum78]. In particular, we will usually label the simple Lie algebras by the Cartan labelling  $(A_l, B_l, \text{etc.}, \text{where } l \text{ is the rank})$  and use the numbering of the simple roots as given on page 58 of [Hum78].

The adjoint action of  $\mathfrak{g}$  on itself—that is, via the bracket—can be extended naturally to tensor products as follows. For  $x, y, z \in \mathfrak{g}$ ,

$$\text{ad}_x(y \otimes z) = \text{ad}_x(y) \otimes z + y \otimes \text{ad}_x(z).$$

We will use this throughout without further comment. We use the term ad-invariant in the obvious way.

We also assume knowledge of the highest weight theory of representations. Our notation for highest-weight  $\mathfrak{g}$ -modules will be  $V(\lambda; \mathfrak{g})$  for a highest weight  $\lambda$ , unless the algebra in question is clear from the context, when we write  $V(\lambda)$ . We may use a notation for a specific realisation of the module, for example  $S^+$  for a positive spin representation. The notation  $V$  (with no  $\lambda$ ) is reserved for the appropriate natural representation, unless otherwise stated.

## 2.2 The semi-classical setting: Lie bialgebras

We now restrict to a field  $k$  of characteristic not 2.

### 2.2.1 Lie bialgebras

The definition of a Lie bialgebra is originally due to Drinfel'd ([Dri83], [Dri87]). The idea is the same as that for Hopf algebras, where we have two structures dual to each other, compatible in a natural way. It is worth commenting that Lie bialgebras form a richer class than Lie algebras: the choice of the cobracket, the dual structure to the bracket, is not usually unique.

**Definition 2.2.1** ([Dri83]). *A Lie bialgebra is  $(\mathfrak{g}, [\ , \ ], \delta)$  where*

- i)  $(\mathfrak{g}, [\ , \ ])$  is a Lie algebra,*
- ii)  $(\mathfrak{g}, \delta)$  is a Lie coalgebra, that is,  $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  satisfies*

$$\delta + \tau \circ \delta = 0 \quad (\text{anticocommutativity})$$

$$(\delta \otimes \text{id}) \circ \delta + \text{cyclic} = 0 \quad (\text{co-Jacobi identity})$$

*(Here, “cyclic” refers to cyclical rotations of the three tensor product factors in  $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ .)*

- iii) we have a cohomological compatibility condition:  $\delta$  is a 1-cocycle in  $Z_{\text{ad}}^1(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g})$ . Explicitly,*

$$\delta([x, y]) = \text{ad}_x(\delta y) - \text{ad}_y(\delta x).$$

Examining this definition, we see that if  $\mathfrak{g}$  is a finite-dimensional Lie bialgebra, then  $(\mathfrak{g}^*, \delta^*, [\ , \ ]^*)$  is also a finite-dimensional Lie bialgebra. Here,  $\delta^*$  and  $[\ , \ ]^*$  are the bracket and cobracket, respectively, given by dualisation.

In many of the natural cases we wish to consider, the cobracket  $\delta$  arises as the coboundary of an element  $r \in \mathfrak{g} \otimes \mathfrak{g}$ . Explicitly,  $\delta x = \text{ad}_x(r)$  for all  $x \in \mathfrak{g}$ . Imposing the further conditions that  $r$  satisfies the classical Yang–Baxter equation and has ad-invariant symmetric part, we say that  $(\mathfrak{g}, r)$  is a quasitriangular Lie bialgebra.

The classical Yang–Baxter equation, in the Lie setting, is

$$[[r, r]] \stackrel{\text{def}}{=} [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

Here, writing  $r = r^{(1)} \otimes r^{(2)}$ , we have  $r_{12} = r^{(1)} \otimes r^{(2)} \otimes 1$ , etc., with the indices showing the placement in the triple tensor product  $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ . The bracket is taken in the common factor, e.g.  $[r_{12}, r_{13}] = [r^{(1)}, r'^{(1)}] \otimes r^{(2)} \otimes r'^{(2)}$  with  $r'$  a second copy of  $r$ . The bracket  $[[ , ]]$  is the Schouten bracket, the natural extension of the bracket to these tensor spaces. We can take  $[[r, s]]$  in the above definition by replacing  $r'$  with  $s$ .

Note that to construct a quasitriangular Lie bialgebra, it is sufficient to find an element  $r \in \mathfrak{g} \otimes \mathfrak{g}$  satisfying the classical Yang–Baxter equation and with ad-invariant symmetric part. Then we take the coboundary  $dr$  for  $\delta$ . Also, any quasitriangular structure  $r$  gives a second for free: it is easy to check that if  $r$  satisfies the conditions above then so does  $-r_{21}$ . We call  $-r_{21}$  the opposite (or conjugate) quasitriangular structure to  $r$ .

Considering the symmetric part of  $r$ ,  $2r_+ \stackrel{\text{def}}{=} r + \tau(r)$ , we can distinguish two sub-cases of quasitriangularity. Firstly, if  $2r_+ = 0$  we say  $(\mathfrak{g}, r)$  is triangular. Secondly, considering  $2r_+$  as a map  $\mathfrak{g}^* \rightarrow \mathfrak{g}$ , if this map is surjective, we say  $(\mathfrak{g}, r)$  is factorisable. We will use “factorisable” and “triangular” for “factorisable quasitriangular” and “triangular quasitriangular”, respectively. We also refer the reader to the paper of Reshetikhin and Semenov-Tyan-Shanskii ([RSTS88]).

For each finite-dimensional complex semisimple Lie algebra  $\mathfrak{g}$ , there exists a canonical quasitriangular structure, the Drinfel’d–Sklyanin solution:

$$r = \sum_{\alpha \in R^+} c_\alpha (X_\alpha^+ \otimes X_\alpha^-) + \frac{1}{2} \sum_{i,j} c_i (C^{-1})_{ij} (H_i \otimes H_j) \quad (2.1)$$

with  $R^+$  the set of positive roots of  $\mathfrak{g}$ ,  $c_\alpha = \frac{\alpha \cdot \alpha}{2}$  and  $C$  the Cartan matrix. In Chapter 3, we concentrate on the Lie algebra structure rather than the coalgebra structure and, unless otherwise stated, we use this choice.

### 2.2.2 Braided-Lie bialgebras

We now consider the braided version of Lie bialgebras, as defined by Majid in [Maj00]. Here we consider the module category  ${}_{\mathfrak{g}}\mathcal{M}$  of a quasitriangular Lie bialgebra  $\mathfrak{g}$  and objects in this category possessing a  $\mathfrak{g}$ -covariant Lie algebra structure. Following the line suggested by the theory of braided groups, we associate to these objects a braiding-type map generalising the usual flip. If  $\mathfrak{b}$  is a  $\mathfrak{g}$ -covariant Lie algebra in the category  ${}_{\mathfrak{g}}\mathcal{M}$ , we define the infinitesimal braiding of  $\mathfrak{b}$  to be the operator  $\psi : \mathfrak{b} \otimes \mathfrak{b} \rightarrow \mathfrak{b} \otimes \mathfrak{b}$ ,  $\psi(a \otimes b) = 2r_+ \triangleright (a \otimes b - b \otimes a)$  where  $\triangleright$  is the left action of  $\mathfrak{g}$  on  $\mathfrak{b}$ . In fact,  $\psi$  is a 2-cocycle in  $Z_{\text{ad}}^2(\mathfrak{b}, \mathfrak{b} \otimes \mathfrak{b})$ .

**Definition 2.2.2** ([Maj00]). *A braided-Lie bialgebra  $(\mathfrak{b}, [\ , \ ]_{\mathfrak{b}}, \underline{\delta})$  is an object in  ${}_{\mathfrak{g}}\mathcal{M}$  satisfying the following conditions.*

- i)  $(\mathfrak{b}, [\ , \ ]_{\mathfrak{b}})$  is a  $\mathfrak{g}$ -covariant Lie algebra in the category.
- ii)  $(\mathfrak{b}, \underline{\delta})$  is a  $\mathfrak{g}$ -covariant Lie coalgebra in the category.
- iii)  $d\underline{\delta} = \psi$ .

There is a braided structure  $\underline{\mathfrak{g}}$  naturally associated to any quasitriangular Lie bialgebra  $\mathfrak{g}$ , coming from the adjoint representation.

**Definition 2.2.3** ([Maj00]). *Take  $\underline{\mathfrak{g}}$  to be the adjoint representation of  $\mathfrak{g}$ . For the Lie bracket of  $\underline{\mathfrak{g}}$ , we take the Lie bracket of  $\mathfrak{g}$ . Clearly, this is covariant. For the braided-cobracket we take*

$$\underline{\delta}x = 2r_+^{(1)} \otimes [x, r_+^{(2)}]$$

*for all  $x \in \underline{\mathfrak{g}}$ . Here,  $r$  is considered as an element of  $\underline{\mathfrak{g}} \otimes \underline{\mathfrak{g}}$  in the obvious way. We call  $\underline{\mathfrak{g}}$  the transmutation of  $\mathfrak{g}$ .*

For a finite-dimensional factorisable Lie bialgebra  $\mathfrak{g}$ , this braided structure is very natural. Using the isomorphism  $2r_+ : \mathfrak{g}^* \rightarrow \mathfrak{g}$  provided by the factorisability assumption, we have that  $\underline{\delta}$  is equivalent to the Kirillov–Kostant Lie cobracket, which is the cobracket given precisely by dualising the Lie bracket of  $\mathfrak{g}$ . Then we see that  $\underline{\mathfrak{g}}$  is self-dual.

This example will be particularly important in Chapter 5 but we will see other examples not of this type in Chapter 3.

### 2.2.3 The Drinfel'd double of a Lie bialgebra

The Drinfel'd double is one of the most studied and most used Lie bialgebra constructions and it will play an equally important role in our study of the triple.

**Definition 2.2.4** ([Dri87]). *Let  $\mathfrak{g}$  be a finite-dimensional Lie bialgebra. The Drinfel'd double,  $D(\mathfrak{g})$ , is the quasitriangular Lie bialgebra given by*

- i)  $\mathfrak{g} \oplus \mathfrak{g}^*$  as base vector space,
- ii) Lie bracket given by  $\mathfrak{g}$  as a sub-Lie algebra in the first part,  $\mathfrak{g}^{*\text{op}}$  as a sub-Lie algebra in the second part and bracket between the two given by

$$[x, \varphi]_D = \varphi_{(1)} \langle \varphi_{(2)}, x \rangle + x_{(1)} \langle \varphi, x_{(2)} \rangle$$

for  $x \in \mathfrak{g}$ ,  $\varphi \in \mathfrak{g}^{*\text{op}}$  and

- iii) Lie cobracket given by the direct sum cobracket, i.e.  $\delta_D x = \delta x$  and  $\delta_D \varphi = \delta^* \varphi$  where  $\delta$  and  $\delta^*$  are the cobrackets on  $\mathfrak{g}$  and  $\mathfrak{g}^*$  respectively.

Note that the above coalgebra structure is given by taking  $r = \sum f^a \otimes e_a$  with  $\{e_a\}$  a basis of  $\mathfrak{g}$  and  $\{f^a\}$  a dual basis. We see that the double is always quasitriangular.

We have a number of different realisations of the double by other constructions. Firstly, we have the double cross sum Lie bialgebra,  $\mathfrak{g} \bowtie \mathfrak{g}^{*\text{op}}$ . Here the two parts act on each other and the cross bracket is obtained from this: we think of a simultaneous two-way semidirect sum. The coalgebra is a direct sum. The actions here are mutual coadjoint actions, namely  $\varphi \triangleright x = x_{(1)} \langle \varphi, x_{(2)} \rangle$  and  $\varphi \triangleleft x = \varphi_{(1)} \langle \varphi_{(2)}, x \rangle$ . Then the double cross sum has bracket

$$[x \oplus \varphi, y \oplus \psi] = ([x, y] + \varphi \triangleright y - \psi \triangleright x) \oplus ([\varphi, \psi] + \varphi \triangleleft y - \psi \triangleleft x)$$

We refer the reader to [Maj95, Section 8.3] for more details and references. A second is given by considering a twisted structure, as described below. We will recall a third, the most relevant as the inspiration for the definition of the triple, at the start of Section 5.1.

We can obtain many non-trivial structures by twisting simple ones using cohomology. In our setting of quasitriangular Lie bialgebras we will only focus on twisting cobrackets, or equivalently, quasitriangular structures. To twist a quasitriangular structure  $r$ , we replace  $r$  by  $r + \chi$ , where  $\chi \in \mathfrak{g} \otimes \mathfrak{g}$  satisfies  $[[r, \chi]] + [[\chi, r]] + [[\chi, \chi]] = 0$  and for all  $\xi \in \mathfrak{g}$ ,  $\text{ad}_\xi(\chi + \chi_{21}) = 0$ . Here,  $[[\ , \ ]]$  is the Schouten bracket as described above. One can check that  $(\mathfrak{g}, [\ , \ ], r + \chi)$  is indeed again a quasitriangular Lie bialgebra. We remark that the first of these conditions is equivalent to

$$\text{ad}_\xi((\text{id} \otimes \delta)\chi + \text{cyclic} + [[\chi, \chi]]) = 0$$

for all  $\xi \in \mathfrak{g}$ . So, in particular, we can satisfy the twisting requirements by choosing  $\chi$  such that  $\chi$  is symmetric and has  $(\text{id} \otimes \delta)\chi + \text{cyclic} + [[\chi, \chi]] = 0$ .

Then we can describe the Drinfel'd double as a twisting, as follows.

**Theorem 2.2.1** ([STS83]). *Let  $(\mathfrak{g}, [\ , \ ], r)$  be a quasitriangular Lie bialgebra. There is a quasitriangular Lie bialgebra  $\mathfrak{g} \blacktriangleright \mathfrak{g}$  given by twisting  $\mathfrak{g} \oplus \mathfrak{g}$  by*

$$\chi = r_{LR} - \tau(r_{LR}) = (r^{(1)} \oplus 0) \otimes (0 \oplus r^{(2)}) - (0 \oplus r^{(2)}) \otimes (r^{(1)} \oplus 0).$$

*Moreover, there is a homomorphism  $D(\mathfrak{g}) \rightarrow \mathfrak{g} \blacktriangleright \mathfrak{g}$  of Lie bialgebras which is an isomorphism when  $\mathfrak{g}$  is factorisable.*  $\square$

The quasitriangular structure on  $\mathfrak{g} \oplus \mathfrak{g}$  we start with in this theorem is  $r \oplus -r_{21}$ , i.e. the direct sum, taking the opposite quasitriangular structure on the second factor.

## 2.2.4 The bosonisation constructions for Lie bialgebras

We are now in a position to state the theorem defining double-bosonisation for Lie bialgebras. Let  $\mathfrak{g}$  be a quasitriangular Lie bialgebra.

**Theorem 2.2.2** ([Maj00, Theorem 3.10]). *For dually paired braided-Lie bialgebras  $\mathfrak{b}, \mathfrak{c}$  in the category of left  $\mathfrak{g}$ -modules  ${}_{\mathfrak{g}}\mathcal{M}$ , the vector space  $\mathfrak{b} \oplus \mathfrak{g} \oplus \mathfrak{c}$  has a unique Lie bialgebra structure  $\mathfrak{b} \rtimes \mathfrak{g} \ltimes \mathfrak{c}^{\text{op}}$ , the double-bosonisation, such that  $\mathfrak{g}$  is a sub-Lie bialgebra,  $\mathfrak{b}, \mathfrak{c}^{\text{op}}$*

are Lie subalgebras, and

$$\begin{aligned} [\xi, x] &= \xi \triangleright x, \quad [\xi, \varphi] = \xi \triangleright \varphi \\ [x, \varphi] &= x_{(1)} \langle \varphi, x_{(2)} \rangle + \varphi_{(1)} \langle \varphi_{(2)}, x \rangle + 2r_+^{(1)} \langle \varphi, r_+^{(2)} \triangleright x \rangle \\ \delta x &= \underline{\delta} x + r^{(2)} \otimes r^{(1)} \triangleright x - r^{(1)} \triangleright x \otimes r^{(2)} \\ \delta \varphi &= \underline{\delta} \varphi + r^{(2)} \triangleright \varphi \otimes r^{(1)} - r^{(1)} \otimes r^{(2)} \triangleright \varphi \end{aligned}$$

for all  $x \in \mathfrak{b}$ ,  $\xi \in \mathfrak{g}$  and  $\varphi \in \mathfrak{c}$ . Here  $\underline{\delta} x = x_{(1)} \otimes x_{(2)}$ . □

Moreover, the double-bosonisation is always quasitriangular. This is established in the case we will need ( $\mathfrak{c} = \mathfrak{b}^*$ ) by the following proposition.

**Proposition 2.2.3** ([Maj00, Proposition 3.11]). *Let  $\mathfrak{b} \in {}_{\mathfrak{g}}\mathcal{M}$  be a finite-dimensional braided-Lie bialgebra with dual  $\mathfrak{b}^*$ . Then the double-bosonisation  $\mathfrak{b} \rtimes \mathfrak{g} \ltimes \mathfrak{b}^{*\text{op}}$  is quasitriangular with*

$$r^{\text{new}} = r + \sum_a f^a \otimes e_a$$

where  $\{e_a\}$  is a basis of  $\mathfrak{b}$  and  $\{f^a\}$  is a dual basis, and  $r$  is the quasitriangular structure of  $\mathfrak{g}$ . If  $\mathfrak{g}$  is factorisable then so is the double-bosonisation. □

We will occasionally refer to the (single) bosonisation, which is defined as follows.

**Theorem 2.2.4** (cf. [Maj00, Theorem 3.5]). *Let  $\mathfrak{b} \in {}_{\mathfrak{g}}\mathcal{M}$  be a braided-Lie bialgebra. Its bosonisation is the Lie bialgebra  $\mathfrak{b} \rtimes \mathfrak{g}$  with semi-direct sum Lie algebra structure, namely  $\mathfrak{g}$ ,  $\mathfrak{b}$  occurring as Lie subalgebras and  $[\xi, x] = \xi \triangleright x$ ,  $\mathfrak{g}$  occurring as a Lie sub-coalgebra and*

$$\delta x = \underline{\delta} x + r^{(2)} \otimes (r^{(1)} \triangleright x) - (r^{(1)} \triangleright x) \otimes r^{(2)}$$

for all  $\xi \in \mathfrak{g}$  and  $x \in \mathfrak{b}$ . □

Note that the single bosonisation occurs as a sub-Lie bialgebra of the double bosonisation. With the notation of Theorem 2.2.2 above,  $\mathfrak{b} \rtimes \mathfrak{g}$  is the sub-Lie bialgebra of  $\mathfrak{b} \rtimes \mathfrak{g} \ltimes \mathfrak{c}^{\text{op}}$  with base vector space  $\mathfrak{b} \oplus \mathfrak{g}$  and Lie algebra and coalgebra structures as in that theorem.



## 2.3 The quantum setting: Hopf algebras

### 2.3.1 Coalgebras, bialgebras and Hopf algebras

Let  $k$  be a field. Throughout we will write  $\otimes$  for  $\otimes_k$ . Then a unital algebra  $A$  over  $k$  is

- i) a vector space over  $k$  with
- ii) a map  $m : A \otimes A \rightarrow A$  such that  $m \circ (m \otimes \text{id}) = m \circ (\text{id} \otimes m)$  and
- iii) a map  $\eta : k \rightarrow A$  such that  $m \circ (\eta \otimes \text{id}) = m \circ (\text{id} \otimes \eta)$ .

*Note:* We will not consider non-unital algebras and will henceforth use “algebra” to mean “unital algebra”.

A coalgebra  $C$  over  $k$  is

- i) a vector space over  $k$  with
- ii) a map  $\Delta : C \rightarrow C \otimes C$  such that  $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$  and
- iii) a map  $\varepsilon : C \rightarrow k$  such that  $(\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta$ .

A bialgebra  $B$  over  $k$  is

- i) an algebra  $(B, m, \eta)$  and
- ii) a coalgebra  $(B, \Delta, \varepsilon)$  with
- iii)  $\Delta$  and  $\varepsilon$  algebra maps.

A Hopf algebra  $H$  over  $k$  is

- i) a bialgebra  $(H, m, \Delta, \eta, \varepsilon)$  over  $k$  with
- ii) a map  $S : H \rightarrow H$  such that  $m \circ (\text{id} \otimes S) \circ \Delta = \eta \circ \varepsilon = m \circ (S \otimes \text{id}) \circ \Delta$ .

It can be shown that  $S$  is an anti-algebra map (an anti-homomorphism) and an anti-coalgebra map. We call  $\Delta$  the coproduct of  $H$ ,  $\varepsilon$  the counit and  $S$  the antipode.

We can extend the concept of a dual in the finite-dimensional setting to that of a dual pairing, as follows. Two  $k$ -Hopf algebras  $H, H'$  are dually paired by a map  $\langle \cdot, \cdot \rangle : H' \otimes H \rightarrow k$  if

$$\begin{aligned} \langle \varphi\psi, h \rangle &= \langle \varphi \otimes \psi, \Delta(h) \rangle, \\ \langle 1, h \rangle &= \varepsilon(h), \\ \langle \Delta(\varphi), h \otimes g \rangle &= \langle \varphi, hg \rangle, \\ \varepsilon(\varphi) &= \langle \varphi, 1 \rangle \quad \text{and} \\ \langle S\varphi, h \rangle &= \langle \varphi, Sh \rangle \end{aligned}$$

for all  $\varphi, \psi \in H'$  and  $h, g \in H$ . Here  $\langle \cdot, \cdot \rangle$  extends to tensor products pairwise. If  $H$  is finite-dimensional, it is dually paired with the usual dual  $H^*$  by taking  $\langle \cdot, \cdot \rangle = \text{ev}$ . This is the unique possibility in this case.

We will also need the notion of a convolution-invertible map. If  $(C, \Delta, \varepsilon)$  is a coalgebra and  $(A, m, \eta)$  an algebra, then the set of linear maps  $\text{Hom}_k(C, A)$  has the structure of an algebra  $(\text{Hom}_k(C, A), \cdot, I)$  via

$$\begin{aligned} \varphi \cdot \psi &= m \circ (\varphi \otimes \psi) \circ \Delta, \text{ that is, } (\varphi \cdot \psi)(c) = \varphi(c_{(1)})\psi(c_{(2)}) \quad \text{and} \\ I &= \eta \circ \varepsilon, \text{ i.e. } I(c) = \varepsilon(c)1, \end{aligned}$$

where  $c \in C$ . An example is the antipode in a Hopf algebra  $H$ , which is the convolution-inverse in  $\text{Hom}_k(H, H)$  of the identity map. Also, since a dual pairing of Hopf algebras  $\langle \cdot, \cdot \rangle : H' \otimes H \rightarrow k$  is an element of  $\text{Hom}_k(H' \otimes H, k)$ , we may talk about a convolution-invertible pairing, namely a pairing invertible under the convolution product in this algebra.

If  $A$  is a  $k$ -algebra, a left  $A$ -module may be defined as a  $k$ -vector space  $M$  and a map  $\psi : A \otimes M \rightarrow M$  such that  $\psi \circ (\eta \otimes \text{id}) = \omega$ , where  $\omega : k \otimes M \rightarrow M$  is the natural isomorphism, and  $\psi \circ (m \otimes \text{id}) = \psi \circ (\text{id} \otimes \psi)$ . The dual definition, that of a comodule of a coalgebra, is as follows: if  $C$  is a  $k$ -coalgebra, a right  $C$ -comodule is a

$k$ -vector space  $M$  together with a map  $\beta : M \rightarrow M \otimes C$  such that  $(\text{id} \otimes \varepsilon) \circ \beta = \omega^{-1}$  and  $(\beta \otimes \text{id}) \circ \beta = (\text{id} \otimes \Delta) \circ \beta$ . Note that  $C$  is a right  $C$ -comodule with structure map  $\Delta$ .

If  $M$  is a right comodule and  $N \subseteq M$  with  $\beta(N) \subseteq N \otimes C$  then  $N$  is a sub-comodule. The right sub-comodules of  $C$  are the right coideals. The isomorphism theorem for comodules is as usual, by universal algebra arguments.

**Definition 2.3.1** (cf. [Dri87]). *A quasitriangular bialgebra or Hopf algebra is a pair  $(H, \mathcal{R})$  where  $H$  is a bialgebra or Hopf algebra,  $\mathcal{R} \in H \otimes H$  is invertible and satisfies*

$$\begin{aligned} \tau \circ \Delta h &= \mathcal{R}(\Delta h)\mathcal{R}^{-1} \text{ for all } h \in H, \\ (\Delta \otimes \text{id})\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{23} \quad \text{and} \quad (\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \end{aligned}$$

where  $\mathcal{R}_{12} = \mathcal{R} \otimes 1$ ,  $\mathcal{R}_{23} = 1 \otimes \mathcal{R}$  etc.

Importantly, the element  $\mathcal{R}$  satisfies the Yang–Baxter equation, or braid relations:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

Here the subscripts indicate the placement in the triple tensor product, so  $\mathcal{R}_{12} = \mathcal{R} \otimes 1$ ,  $\mathcal{R}_{13} = \mathcal{R}^{(1)} \otimes 1 \otimes \mathcal{R}^{(2)}$  and  $\mathcal{R}_{23} = 1 \otimes \mathcal{R}$ .

As for Lie bialgebras, we have a definition of factorisable quasitriangular. Let  $\mathcal{Q}$  denote the product  $\tau(\mathcal{R})\mathcal{R}$  and let  $\Phi : H^* \rightarrow H$  be the map defined by  $\varphi \mapsto (\varphi \otimes \text{id})(\mathcal{Q})$ . Then  $(H, \mathcal{R})$  is factorisable quasitriangular (or just factorisable) if  $\Phi$  is surjective.

### 2.3.2 Hopf algebras in braided categories

If  $\mathcal{C}$  is a category, let  $\text{Obj}(\mathcal{C})$  denote the class of objects of  $\mathcal{C}$  and  $\text{Mor}_{\mathcal{C}}(A, B)$  the set of morphisms between the objects  $A$  and  $B$  in  $\mathcal{C}$ .

**Definition 2.3.2.** *A monoidal category  $(\mathcal{C}, - \otimes -, \mathbb{1}, \Phi, \mathbb{1} \otimes -, - \otimes \mathbb{1})$  consists of the following data:*

- i) a category  $\mathcal{C}$ ,
- ii) a functor  $- \otimes - : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ ,

iii) a natural isomorphism  $\Phi : (- \otimes -) \otimes - \rightarrow - \otimes (- \otimes -)$ , that is, a collection of functorial isomorphisms

$$\Phi_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$$

for all  $A, B, C \in \mathcal{C}$  satisfying the pentagon identity:

$$\begin{array}{ccccc}
 & & ((A \otimes B) \otimes C) \otimes D & & \\
 & \swarrow \Phi_{A,B,C} \otimes \text{id} & & \searrow \Phi_{A \otimes B, C, D} & \\
 (A \otimes (B \otimes C)) \otimes D & & & & (A \otimes B) \otimes (C \otimes D) \\
 \downarrow \Phi_{A, B \otimes C, D} & & & & \downarrow \Phi_{A, B, C \otimes D} \\
 A \otimes ((B \otimes C) \otimes D) & & & & \\
 \searrow \text{id} \otimes \Phi_{B, C, D} & & & \swarrow \Phi_{A, B, C \otimes D} & \\
 & A \otimes (B \otimes (C \otimes D)) & & & 
 \end{array}$$

iv) a unit object  $\mathbb{1}$ ,

v) natural isomorphisms  $\mathbb{1} \otimes - \cong \text{id}$ ,  $- \otimes \mathbb{1} \cong \text{id}$  satisfying the triangle condition:

$$\begin{array}{ccc}
 & A \otimes B & \\
 (- \otimes \mathbb{1}) \otimes \text{id} \swarrow & & \searrow \text{id} \otimes (\mathbb{1} \otimes -) \\
 (A \otimes \mathbb{1}) \otimes B & \xrightarrow{\Phi_{A, \mathbb{1}, B}} & A \otimes (\mathbb{1} \otimes B)
 \end{array}$$

The pentagon identity together with Mac Lane's coherence theorem allow us to consider a monoidal category to be equivalent to an associative category, in the sense that we may remove the brackets from expressions such as  $((A \otimes B) \otimes C) \otimes D$  and write simply  $A \otimes B \otimes C \otimes D$ .

The prototype example of a monoidal category is the category  $\text{Vec}_k$  of vector spaces over a field  $k$ . The map  $- \otimes -$  is the usual tensor product of vector spaces and the unit object is the field,  $\mathbb{1} = k$ .

Now let  $- \otimes^{\text{op}} -$  denote the functor defined by  $A \otimes^{\text{op}} B = B \otimes A$ .

**Definition 2.3.3** ([JS93]). A braided monoidal category  $(\mathcal{C}, - \otimes -, \mathbb{1}, \Phi, \mathbb{1} \otimes -, - \otimes \mathbb{1}, \Psi)$  consists of the following data:

- i) a monoidal category  $(\mathcal{C}, - \otimes -, \mathbb{1}, \Phi, \mathbb{1} \otimes -, - \otimes \mathbb{1})$ ,
- ii) a natural isomorphism  $\Psi : - \otimes - \rightarrow - \otimes^{\text{op}} -$ , that is, a collection of functorial isomorphisms

$$\Psi_{A,B} : A \otimes B \xrightarrow{\sim} A \otimes^{\text{op}} B = B \otimes A$$

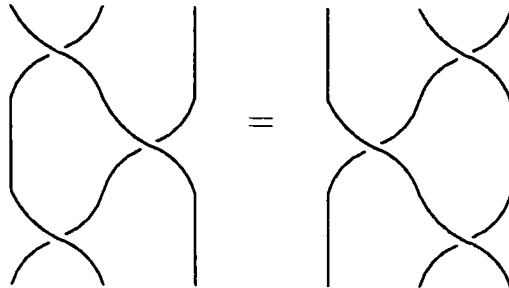
for all  $A, B \in \mathcal{C}$  satisfying the hexagon identities:

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & (A \otimes B) \otimes C & & \\
 & \swarrow \Phi_{A,B,C} & & \searrow \Psi_{A,B} \otimes \text{id} & \\
 A \otimes (B \otimes C) & & & & (B \otimes A) \otimes C \\
 \downarrow \Psi_{A,B \otimes C} & & & & \downarrow \Phi_{B,A,C} \\
 (B \otimes C) \otimes A & & & & B \otimes (A \otimes C) \\
 & \searrow \Phi_{B,C,A} & & \swarrow \text{id} \otimes \Psi_{A,C} & \\
 & & B \otimes (C \otimes A) & & 
 \end{array} \\
 \\
 \begin{array}{ccccc}
 & & A \otimes (B \otimes C) & & \\
 & \swarrow \text{id} \otimes \Psi_{B,C} & & \searrow \Phi_{A,B,C}^{-1} & \\
 A \otimes (C \otimes B) & & & & (A \otimes B) \otimes C \\
 \downarrow \Phi_{A,C,B}^{-1} & & & & \downarrow \Psi_{A \otimes B, C} \\
 (A \otimes C) \otimes B & & & & C \otimes (A \otimes B) \\
 & \searrow \Psi_{A,C} \otimes \text{id} & & \swarrow \Phi_{C,A,B}^{-1} & \\
 & & (C \otimes A) \otimes B & & 
 \end{array}
 \end{array}$$

We will often shorten ‘braided monoidal category’ to just ‘braided category’. The hexagon identities together with the functoriality of  $\Psi$  imply the Yang–Baxter equation:

$$\Psi_{A,B} \circ \Psi_{A,C} \circ \Psi_{B,C} = \Psi_{B,C} \circ \Psi_{A,C} \circ \Psi_{A,B}.$$

This is represented graphically as



The prototype example of a braided category is the module category  ${}_H\mathcal{M}$  of a quasitriangular bialgebra. Denote the left action of  $H$  by  $\triangleright$ . Then the braiding map  $\Psi$  is given by  $\Psi_{V,W}(v \otimes w) = \mathcal{R}^{(2)} \triangleright w \otimes \mathcal{R}^{(1)} \triangleright v$  for  $v \in V$ ,  $w \in W$ , with  $V, W \in {}_H\mathcal{M}$ .

We may consider objects in categories with algebraic structures on them. In generality, this is an object together with some morphisms from the category that satisfy the axioms for the appropriate algebraic structure, when we translate axioms into identities of morphisms. The first example we will need is that of an algebra in a monoidal category  $\mathcal{C}$ . This is a tuple  $(A, m, \eta)$  with  $A \in \text{Obj}(\mathcal{C})$ ,  $m \in \text{Mor}_{\mathcal{C}}(A \otimes A, A)$  and  $\eta \in \text{Mor}_{\mathcal{C}}(\mathbb{1}, A)$  satisfying the identities in the definition of an algebra in Section 2.3.1. Indeed, the definition there is of an algebra in the monoidal category  $\text{Vec}_k$ .

Recall that if  $A$  and  $B$  are algebras, we may give  $A \otimes B$  an algebra structure by  $m_{\otimes} = (m_A \otimes m_B) \circ (\text{id} \otimes \tau \otimes \text{id})$ , where  $m_A, m_B$  are the product maps for  $A$  and  $B$  and  $\tau$  is the tensor product flip map. In a braided category, we replace the (symmetric) flip map  $\tau$  with the truly braided map  $\Psi$ . This defines the braided tensor product algebra  $A \underline{\otimes} B$ , with  $m_{\underline{\otimes}} = (m_A \otimes m_B) \circ (\text{id} \otimes \Psi_{B,A} \otimes \text{id})$ , as an algebra in the braided category.

This allows us to make the following definition.

**Definition 2.3.4** (cf. [Maj94]). *A Hopf algebra in a braided category  $(B, \underline{m}, \underline{\eta}, \underline{\Delta}, \underline{\varepsilon}, \underline{S})$  is*

- i) *an algebra  $(B, \underline{m}, \underline{\eta})$  in the braided category,*
- ii) *a coalgebra  $(B, \underline{\Delta}, \underline{\varepsilon})$  with  $\underline{\Delta} : B \rightarrow B \underline{\otimes} B$  and  $\underline{\varepsilon} : B \rightarrow \mathbb{1}$  morphisms in the category and algebra maps (with respect to the algebra structure on  $B$  and the braided algebra structure on  $B \underline{\otimes} B$ ) and*

iii) a map  $\underline{S} : B \rightarrow B$  such that  $\underline{m} \circ (\text{id} \otimes \underline{S}) \circ \underline{\Delta} = \underline{\eta} \circ \underline{\varepsilon} = \underline{m} \circ (\underline{S} \otimes \text{id}) \circ \underline{\Delta}$  (as morphisms in the category).

We will also use the term ‘braided group’ for a Hopf algebra in a braided category. We use the adjective ‘braided’ to distinguish the Hopf algebra structures in a braided category, e.g.  $\underline{\Delta}$  is the braided-coproduct of  $B$ .

### 2.3.3 The Drinfel’d double of a Hopf algebra

We now define the Drinfel’d double in the form we will require. We use a variant on the usual definition, as the Hopf algebras we will use are not necessarily finite-dimensional. The original reference is [Dri87] but we use a form described in [Maj95, Chapter 7].

**Definition 2.3.5.** *Let  $H$  and  $H'$  be dually paired  $k$ -Hopf algebras, paired by the map  $\langle , \rangle : H \otimes H' \rightarrow k$ . Assume that  $H'$  has an invertible antipode and denote the (induced) convolution-inverse of  $\langle , \rangle$  by  $\langle , \rangle^{-1}$ . Then the Drinfel’d double  $D(H, H')$  in the form  $H \bowtie (H')^{\text{op}}$  is a Hopf algebra, built on the vector space  $H \otimes H'$  with  $H$  and  $(H')^{\text{op}}$  sub-Hopf algebras and cross-relations*

$$\langle h_{(1)}, a_{(1)} \rangle h_{(2)} a_{(2)} = a_{(1)} h_{(1)} \langle h_{(2)}, a_{(2)} \rangle.$$

The product on  $D(H, H')$  is given by

$$(h \otimes a)(g \otimes b) = hg_{(2)} \otimes ba_{(2)} \langle g_{(1)}, a_{(1)} \rangle \langle g_{(3)}, a_{(3)} \rangle^{-1}$$

and  $D(H, H')$  has the tensor product unit, coproduct and counit.

If  $H = H'$  (i.e.  $H$  is self-dually paired) or  $H$  is finite-dimensional so that  $H' = H^*$  is forced, we will write simply  $D(H)$  for  $D(H, H)$  and  $D(H, H^*)$ , respectively. We note in particular that if  $H$  is finite-dimensional, then  $D(H)$  is a factorisable quasitriangular Hopf algebra, with

$$\mathcal{R} = \sum_a (f^a \otimes 1) \otimes (1 \otimes e_a),$$

with  $\{e_a\}$  a basis of  $H$  and  $\{f^a\}$  a dual basis. This is related to the categorical coevaluation.

### 2.3.4 The bosonisation constructions for Hopf algebras

Bosonisation and double-bosonisation are the two key constructions which make this inductive approach to the study of the quantized enveloping algebras possible. Bosonisation takes a Hopf algebra and a braided group in its category of modules and combines these to obtain a new (ordinary) Hopf algebra. Double-bosonisation incorporates the dual of the braided group as well and again produces a Hopf algebra.

However, bosonisation and double-bosonisation require an additional condition on the initial Hopf algebra  $H$  which forms the input into the constructions. This condition is the existence of a weak quasitriangular structure on  $H$  and  $H'$  dually paired to  $H$ . The existence of a weak quasitriangular structure is, as the name suggests, a weaker condition than quasitriangularity.

**Definition 2.3.6** (cf. [Maj90b]). *Let  $H$  and  $H'$  be dually paired  $k$ -Hopf algebras, paired by the map  $\langle \cdot, \cdot \rangle : H \otimes H' \rightarrow k$ . A weak quasitriangular system consists of  $H$ ,  $H'$  and a pair of convolution-invertible algebra and anti-coalgebra maps  $\mathcal{R}, \bar{\mathcal{R}} : H' \rightarrow H$ , with convolution-inverses  $\mathcal{R}^{-1}, \bar{\mathcal{R}}^{-1}$  respectively, such that*

$$i) \quad \langle \psi, \bar{\mathcal{R}}(\varphi) \rangle = \langle \varphi, \mathcal{R}^{-1}(\psi) \rangle \text{ for all } \psi, \varphi \in H' \text{ and}$$

$$ii) \quad \mathcal{R} \text{ and } \bar{\mathcal{R}} \text{ intertwine the left and right coregular actions } L^*, R^* \text{ with respect to the convolution product } \cdot \text{ on } \text{Hom}_k(H', H):$$

$$L^*(h)(a) \stackrel{\text{def}}{=} \langle h_{(1)}, a \rangle h_{(2)}, \quad R^*(h)(a) \stackrel{\text{def}}{=} h_{(1)} \langle h_{(2)}, a \rangle$$

$$R^*(h) = \mathcal{R} \cdot L^*(h) \cdot \mathcal{R}^{-1}$$

$$R^*(h) = \bar{\mathcal{R}} \cdot L^*(h) \cdot \bar{\mathcal{R}}^{-1}$$

where we consider  $L^* : H' \otimes H \rightarrow H$  as a map  $L^*(h) : H' \rightarrow H$  by fixing  $h \in H$  (similarly for  $R^*$ ).

We will denote by  $WQ(H, H', \mathcal{R}, \bar{\mathcal{R}})$  a weak quasitriangular system with the above data.

We may also interpret  $L^*$  and  $R^*$  as left and right differentiation operators. We can now define the bosonisation construction.



**Definition 2.3.7.** Let  $WQ(H, H', \mathcal{R}, \bar{\mathcal{R}})$  be a weak quasitriangular system and let  $B$  be a Hopf algebra in the braided category of right  $H'$ -comodules,  $\mathcal{M}^{H'}$ . Let  $\beta : B \rightarrow B \otimes H'$ ,  $\beta(b) = b^{(1)} \otimes b^{(2)}$ , denote the right coaction. Then the bosonisation of  $B$ , denoted  $B \bowtie H$ , is the Hopf algebra with

- i) underlying vector space  $B \otimes H$ ,
- ii) semi-direct product by the action  $\triangleright$  given by evaluation against the right coaction of  $H'$ :

$$\begin{aligned} (b \otimes h)(c \otimes g) &= b(h_{(1)} \triangleright c) \otimes h_{(2)}g \\ h \triangleright b &= b^{(1)} \langle h, b^{(2)} \rangle \quad \forall h \in H, b \in B \end{aligned}$$

- iii) semi-direct coproduct by the coaction  $\alpha$  of  $H$  induced by the right coaction of  $H'$  and the weak quasitriangular structure:

$$\Delta(b \otimes h) = \underline{b_{(1)}} \otimes \mathcal{R}(\underline{b_{(2)}}^{(2)})h_{(1)} \otimes \underline{b_{(2)}}^{(1)} \otimes h_{(2)}$$

- iv) tensor product unit and counit and
- v) an antipode, given by an explicit formula which we omit.

In fact, this is the ‘left-handed’ version of bosonisation. The ‘right-handed’ version is given as follows:

**Definition 2.3.8.** Let  $WQ(H, H', \mathcal{R}, \bar{\mathcal{R}})$  be a weak quasitriangular system and let  $B$  be a Hopf algebra in the braided category of left  $H'$ -comodules,  ${}^{H'}\mathcal{M}$ . Let  $\beta : B \rightarrow H' \otimes B$ ,  $\beta(b) = b^{(1)} \otimes b^{(2)}$ , denote the left coaction. Then the right bosonisation of  $B$ , denoted  $H \bowtie B$ , is the Hopf algebra with

- i) underlying vector space  $H \otimes B$ ,
- ii) semi-direct product by the action  $\triangleleft$  given by evaluation against the left coaction of  $H'$ :

$$\begin{aligned} (h \otimes b)(g \otimes c) &= hg_{(1)} \otimes (b \triangleleft g_{(2)})c \\ b \triangleleft h &= \langle h, b^{(1)} \rangle b^{(2)} \quad \forall h \in H, b \in B \end{aligned}$$

- iii) *semi-direct coproduct by the coaction  $\alpha$  of  $H$  induced by the left coaction of  $H'$  and the weak quasitriangular structure:*

$$\Delta(h \otimes b) = h_{(1)} \otimes \underline{b_{(1)}}^{(1)} \otimes h_{(2)} \mathcal{R}(\underline{b_{(1)}}^{(2)}) \otimes \underline{b_{(2)}}$$

- iv) *tensor product unit and counit and*

- v) *an antipode, given by an explicit formula which we again omit.*

Double-bosonisation is defined by combining a left and a right bosonisation, with some cross relations.

**Definition 2.3.9** (cf. [Maj99]). *Let  $WQ(H, H', \mathcal{R}, \bar{\mathcal{R}})$  be a weak quasitriangular system and let  $B$  be a Hopf algebra in the braided category of right  $H'$ -comodules,  $\mathcal{M}^{H'}$ . Let  $B'$  be another Hopf algebra in the braided category  $\mathcal{M}^{H'}$  with an invertible braided-antipode and let  $\text{ev} : B \otimes B' \rightarrow k$  be a dual pairing of the Hopf algebra structures in this category. Then the double-bosonisation  $B \rtimes H \ltimes (B')^{\text{op}}$  of  $B$  and  $(B')^{\text{op}}$  by  $H$  is the Hopf algebra with*

- i) *underlying vector space  $B \otimes H \otimes B'$ ,*
- ii) *sub-Hopf algebras  $B \rtimes H (\equiv B \rtimes H \otimes 1)$ ,  $H \ltimes (B')^{\text{op}}$ , and*
- iii) *cross-relations*

$$\underline{b_{(1)}} \mathcal{R}(\underline{b_{(2)}}^{(2)}) \underline{c_{(1)}} \text{ev}(\underline{c_{(2)}} \otimes \underline{b_{(2)}}^{(1)}) = \text{ev}(\underline{c_{(1)}} \otimes \underline{b_{(1)}}^{(2)}) \underline{c_{(2)}} \bar{\mathcal{R}}(\underline{b_{(1)}}^{(1)}) \underline{b_{(2)}}$$

*for all  $b \in B$ ,  $c \in (B')^{\text{op}}$ .*

### 2.3.5 Pull-backs and push-outs of actions and coactions

We recall the following basic facts from the representation theory of algebras. We will need these in Chapter 4, where we will want to examine the restriction of Hopf algebra representations and also pass to certain quotient algebras. In addition, the dual picture will be needed, that is, the corresponding results for comodules.

**Proposition 2.3.1.** *Let  $A$  be a  $k$ -algebra and  $M$  a (left)  $A$ -module, via the action  $\triangleright : A \otimes M \rightarrow M$ . For  $I \trianglelefteq A$  an ideal of  $A$ , consider the exact sequence of algebras*

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\pi} A/I \longrightarrow 0 . \quad (2.2)$$

*Then:*

*i)  $M$  is an  $I$ -module via the pull-back of  $\triangleright$  along  $i$ :*

$$\triangleright \circ (i \otimes \text{id}) = \triangleright_I : I \otimes M \rightarrow A \otimes M \rightarrow M.$$

*ii) The sequence (2.2) splits: choose  $R$  a set of  $I$ -coset representatives,  $R = \{r_i\}$ . Define  $k_R : A/I \rightarrow A$  by  $a + I \mapsto r_i$  if  $a + I = r_i + I$ . Then  $k_R$  is an algebra map and  $\pi \circ k_R = \text{id}_{A/I}$ .*

*If the pull-back of  $\triangleright$  along  $i$ ,  $\triangleright_I = \triangleright \circ (i \otimes \text{id})$ , is zero then  $M$  is an  $A/I$ -module via the pull-back of  $\triangleright$  along  $k_R$ :*

$$\triangleright \circ (k_R \otimes \text{id}) = \triangleright_{A/I} : A/I \otimes M \rightarrow A \otimes M \rightarrow M. \quad \square$$

We have rephrased these results somewhat from their usual presentation. In particular, the condition in *ii)* that the pull-back of  $\triangleright$  along  $i$  is zero is more usually expressed as “ $I$  annihilates  $M$ ”, i.e. for all  $j \in I$ ,  $m \in M$ ,  $j \triangleright m = 0$ . We have used this formulation to ease the dualisation of this proposition to the coalgebra setting, which is as follows:

**Proposition 2.3.2.** *Let  $C$  be a  $k$ -coalgebra and  $M$  a (left)  $C$ -comodule, via the coaction  $\beta : M \rightarrow C \otimes M$ . For  $D \trianglelefteq C$  a coideal of  $C$  (that is,  $\Delta D \subseteq D \otimes C + C \otimes D$ ), consider the exact sequence of coalgebras*

$$0 \longrightarrow D \xrightarrow{i} C \xrightarrow{\pi} C/D \longrightarrow 0 . \quad (2.3)$$

*Then:*

*i)  $M$  is a  $C/D$ -comodule via the push-out of  $\beta$  along  $\pi$ :*

$$(\pi \otimes \text{id}) \circ \beta = \beta_{C/D} : M \rightarrow C \otimes M \rightarrow C/D \otimes M.$$

*ii) Assume that the sequence (2.3) splits, with a splitting coalgebra map  $j : C/D \rightarrow C$  such that  $j \circ i = \text{id}_D$ . Then  $M$  is a  $D$ -comodule via the push-out of  $\beta$  along  $j$ :*

$$(j \otimes \text{id}) \circ \beta = \beta_D : M \rightarrow C \otimes M \rightarrow D \otimes M. \quad \square$$

### 2.3.6 Quantized enveloping algebras and their representations

Let  $k$  be a field. Recall from Section 2.1.1 the definition of a root datum: let

$$\mathfrak{T} = (I, \cdot, Y, X, i_1 : I \hookrightarrow Y, i_2 : I \hookrightarrow X)$$

be a root datum. Recall also that we set  $c_i = \frac{i \cdot i}{2}$ . Fix  $q \in k^*$  such that  $q^{2c_i} \neq 1$  for all  $i \in I$  and let  $q_i \stackrel{\text{def}}{=} q^{c_i}$ .

We will use the symmetric  $q$ -integers throughout. These, and the corresponding  $q$ -factorial and  $q$ -binomial are defined as follows:

$$[a]_i = \frac{q_i^a - q_i^{-a}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{j=1}^n [j]_i, \quad \begin{bmatrix} n \\ k \end{bmatrix}_i = \frac{[n]_i!}{[k]_i! [n-k]_i!}.$$

Let the identity element of  $Y$  be denoted 0 and let  $Z$  denote the free Abelian subgroup  $\mathbb{Z}[i_1(I)]$  of  $Y$ . We can now define the quantized enveloping algebra  $U_q(\mathfrak{T})$  associated to the root datum  $\mathfrak{T}$  over the field  $k$  with deformation parameter  $q$ .

**Definition 2.3.10.** We define  $U_q(\mathfrak{T})$  to be the Hopf algebra over  $k$  generated by  $E_i, F_i$  ( $i \in I$ ) and  $K_\mu$  ( $\mu \in Z$ ), subject to relations

$$(R1) \quad K_0 = 1, \quad K_\mu K_\nu = K_{\mu+\nu}$$

$$(R2) \quad K_\mu E_i = q^{<\mu, i_2(i)>} E_i K_\mu$$

$$(R3) \quad K_\mu F_i = q^{-<\mu, i_2(i)>} F_i K_\mu$$

$$(R4) \quad E_i F_j - F_j E_i = \delta_{ij} \frac{H_i - H_i^{-1}}{q_i - q_i^{-1}}, \text{ where } H_i \stackrel{\text{def}}{=} K_i^{c_i}$$

$$(R5) \quad \sum_{m=0}^{1-C_{ij}} (-1)^m \begin{bmatrix} 1-C_{ij} \\ m \end{bmatrix}_i E_i^{1-C_{ij}-m} E_j E_i^m = 0$$

$$(R6) \quad \sum_{m=0}^{1-C_{ij}} (-1)^m \begin{bmatrix} 1-C_{ij} \\ m \end{bmatrix}_i F_i^{1-C_{ij}-m} F_j F_i^m = 0$$

The Hopf structure is:

$$\begin{array}{lll} \Delta E_i = E_i \otimes 1 + H_i \otimes E_i & \varepsilon(E_i) = 0 & S E_i = -H_i^{-1} E_i \\ \Delta F_i = F_i \otimes H_i^{-1} + 1 \otimes F_i & \varepsilon(F_i) = 0 & S F_i = -F_i H_i \\ \Delta K_\mu = K_\mu \otimes K_\mu & \varepsilon(K_\mu) = 1 & S K_\mu = K_\mu^{-1} \end{array}$$

*Note:*

- i) There are several definitions of the quantized enveloping algebras in the literature and this one is a slight modification of that of Lusztig ([Lus93]), who has generators  $K_\mu$  with  $\mu \in Y$ . Our definition resembles that of Jantzen in [Jan96], although he starts with root systems, rather than root data. The reason for the restriction to generators indexed by the subgroup  $Z$  rather than  $Y$  is technical and is discussed below.
- ii) By relation (R1),  $K_{-i} = K_i^{-1}$  and we will usually write the latter.
- iii)  $Z$  is finitely generated and we could define  $U_q(\mathfrak{T})$  using only  $K_i, i \in I$ . So  $U_q(\mathfrak{T})$  is in fact finitely generated, rather than infinitely generated as it may first appear.

We will also need certain subalgebras of  $U_q(\mathfrak{T})$ , generated by certain subsets of the generating set for  $U_q(\mathfrak{T})$ , as follows:

$$U_q^0(\mathfrak{T}) = \langle K_\mu \mid \mu \in Z \rangle$$

$$U_q^+(\mathfrak{T}) = \langle E_i \mid i \in I \rangle$$

$$U_q^-(\mathfrak{T}) = \langle F_i \mid i \in I \rangle$$

$$U_q^\geq(\mathfrak{T}) = \langle E_i, K_\mu \mid i \in I, \mu \in Z \rangle$$

$$U_q^\leq(\mathfrak{T}) = \langle F_i, K_\mu \mid i \in I, \mu \in Z \rangle$$

These subalgebras are the quantized enveloping algebra analogues of the Cartan subalgebra, subalgebras of positive and negative root vectors and the positive and negative Borel subalgebras, respectively.

Unfortunately,  $U_q(\mathfrak{T})$  is not a quasitriangular Hopf algebra in general. This is because the analogue of the Drinfel'd-Sklyanin quasitriangular structure for Lie bialgebras (2.1) involves an infinite sum, since  $U_q(\mathfrak{T})$  is infinite-dimensional. There are several approaches to resolving this problem. Drinfel'd ([Dri87]) works in the setting of formal power series in a deformation parameter; Lusztig ([Lus93, Chapter 4]) introduces a topological completion. However, the notion of weak quasitriangularity (Definition 2.3.6) was introduced by Majid in order to avoid these and remain in a purely algebraic setting.

In the context of constructing  $U_q(\mathfrak{T})$  as a double-bosonisation starting from the Hopf algebra  $U_q^0(\mathfrak{T}) = k[Z]$  (the group algebra of  $Z$ ), it follows from [Maj02, Proposition 18.7] that we have a weak quasitriangular system  $WQ(k[Z], k\mathbb{Z}[i_2(I)], \mathcal{R}, \bar{\mathcal{R}})$ , as follows. Let  $\{h_i \mid i \in I\}$  be a basis of  $k\mathbb{Z}[i_2(I)]$ . Then  $\mathcal{R}(h_i) = H_i$  and  $\bar{\mathcal{R}}(h_i) = H_i^{-1}$ . To extend this to the whole of  $U_q(\mathfrak{T})$ , we use Lusztig's pairing between  $U_q^+(\mathfrak{T})$  and  $U_q^-(\mathfrak{T})$ , given by  $(E_i, F_j) = (q_i^{-1} - q_i)^{-1} \delta_{ij}$ . This induces dual bases  $\{f^a\}$  and  $\{e_a\}$  and we have the quasi- $\mathcal{R}$ -matrix, i.e. the formal series  $\sum_a f^a \otimes e_a$ . Then the  $\mathcal{R}, \bar{\mathcal{R}}$  are given by appropriate evaluations against the pairing  $(\ , \ )$  and this is well-defined. We will not give explicit formulæ here.

Similarly, we obtain a self-duality pairing of  $U_q^\leq(\mathfrak{T})$ , as follows. For  $i, j \in I$ , define

$$\begin{aligned} (K_i, K_j) &= q^{<i_1(i), i_2(j)>}, \\ (F_i, F_j) &= -(q_i - q_i^{-1})^{-1} \delta_{ij} \quad \text{and} \\ (K_i, F_j) &= (F_j, K_i) = 0, \end{aligned}$$

extended to the whole of  $U_q^\leq(\mathfrak{T}) \otimes U_q^\leq(\mathfrak{T})$ . One proof that this is a dual pairing of Hopf algebras is in [Jan96, Chapter 6], where the pairing is expressed as a pairing of  $U_q^\leq(\mathfrak{T})$  with  $U_q^\geq(\mathfrak{T})^{\text{op}}$ —we recall that we may identify  $U_q^\geq(\mathfrak{T})^{\text{op}}$  with  $U_q^\leq(\mathfrak{T})$ . As Jantzen observes, the idea goes back to Drinfel'd. It is in order to have this pairing that we index the generators of  $U_q^0(\mathfrak{T})$  by  $Z$  rather than  $Y$ .

Then we may construct the double of  $U_q^\leq(\mathfrak{T})$ ,  $D(U_q^\leq(\mathfrak{T}))$ , as in Definition 2.3.3. As a double cross product, this is  $U_q^\leq(\mathfrak{T}) \bowtie U_q^\geq(\mathfrak{T})$ . Now following Drinfel'd we may recover  $U_q(\mathfrak{T})$  as a quotient of  $D(U_q^\leq(\mathfrak{T}))$ . Observe that  $D(U_q^\leq(\mathfrak{T}))$  is generated by  $\{F_i \otimes 1, 1 \otimes E_i \mid i \in I\} \cup \{K_\mu \otimes 1, 1 \otimes K_\mu \mid \mu \in Z\}$ . Then the quotient  $U_q(\mathfrak{T})$  is obtained by identifying the two Cartan parts, i.e. we impose the relation  $K_\mu \otimes 1 = 1 \otimes K_\mu$ . The corresponding ideal defining the quotient is generated by elements of the form  $K_\mu \otimes K_\mu^{-1} - 1 \otimes 1$ . We will refer to the projection  $\mathbb{P} : D(U_q^\leq(\mathfrak{T})) \rightarrow U_q(\mathfrak{T})$  as Drinfel'd's projection.

We will not consider all representations of  $U_q(\mathfrak{T})$  but as usual concentrate on those modules that decompose into weight spaces.

**Definition 2.3.11** (cf. [Lus93, §3.4.1]). If  $\mathfrak{T} = (I, \cdot, Y, X, \langle \cdot, \cdot \rangle, i_1, i_2)$  is a root datum, a (left)  $U_q(\mathfrak{T})$ -module  $M$  is said to be a weight module if it is the direct sum of its weight spaces. The weight space  $M^\lambda$  associated to  $\lambda \in X$  is defined as

$$M^\lambda = \{m \in M \mid K_\mu \triangleright m = q^{\langle \mu, \lambda \rangle} m \text{ for all } \mu \in Z\}$$

where  $\triangleright$  denotes the left action.

We may rephrase this in terms of coactions. We have a duality pairing of group Hopf algebras  $\ll \cdot, \cdot \gg : k[Z] \otimes k[X] \rightarrow k$  given by  $\ll K_\mu, L_\lambda \gg = q^{\langle \mu, \lambda \rangle}$  for all  $\mu \in Z, \lambda \in X$ . Here,  $\{L_\lambda \mid \lambda \in X\}$  is a basis for  $k[X]$ . So we may write

$$M^\lambda = \{m \in M \mid K_\mu \triangleright m = \ll K_\mu, L_\lambda \gg m \text{ for all } \mu \in Z\}.$$

That is,  $m$  has eigenvalue  $\ll K_\mu, L_\lambda \gg$  for the action of  $K_\mu$ .

Further, we have the coaction  $\beta : M \rightarrow M \otimes k[X]$ ,  $\beta(m) = \sum_\lambda m_\lambda \otimes L_\lambda$ , where  $m = \sum_\lambda m_\lambda$  is the (unique) decomposition of  $m$  with  $m_\lambda \in M^\lambda$ , which exists since  $M = \bigoplus_\lambda M^\lambda$ . This coaction is dual to the action of  $k[Z] \equiv U_q^0(\mathfrak{T})$ : if  $\beta(m) = m^{(1)} \otimes m^{(2)}$ , then  $K_\mu \triangleright m = \ll K_\mu, m^{(2)} \gg m^{(1)}$ . The weight space  $M^\lambda$  is then the set of  $m \in M$  with  $\beta(m) = m \otimes L_\lambda$ . We denote the category of  $U_q(\mathfrak{T})$ -weight modules by  ${}_{\mathfrak{T}}\mathcal{M}^{\text{wt}}$ . If  $M = \bigoplus_{\lambda \in X} M^\lambda$  is a  $U_q(\mathfrak{T})$ -weight module, let  $P_{\mathfrak{T}}(M) = \{\lambda \in X \mid M^\lambda \neq 0\}$  be the set of weights of  $M$ .

Among the set of weights of a module, the dominant weights are particularly important. We extend the definition of dominant in [Lus93, §3.5.5] slightly, as we will need to consider weights and their properties with respect to more than one quantized enveloping algebra. So we define dominance relative to certain subsets of  $I$ , namely those whose image in the cocharacter lattice is linearly independent. As noted by Lusztig, one can define dominance without this linear independence but it is of no use.

**Definition 2.3.12.** Let  $\mathfrak{T} = (I, \cdot, Y, X, \langle \cdot, \cdot \rangle, i_1, i_2)$  be a root datum. For  $\lambda \in X$  and any subset  $S \subseteq I$  such that the set  $\{i_1(s) \mid s \in S\}$  is linearly independent in  $Y$ , we say  $\lambda$  is  $S$ -dominant if  $\langle i_1(s), \lambda \rangle \in \mathbb{N}$  for all  $s \in S$ . Denote the set of  $S$ -dominant  $\lambda \in X$  by  $D_S^{\mathfrak{T}}(X)$ .

For any weight  $\lambda$ , we have two important modules, the Verma module  $M(\lambda)$  and the Weyl module  $L(\lambda)$  of highest weight  $\lambda$ . For brevity, write  $U$  for  $U_q(\mathfrak{T})$  and fix  $\lambda \in X$ . Then, following Jantzen ([Jan96]), we first define the left ideal

$$J_\lambda = \sum_{i \in I} U E_i + \sum_{i \in I} U (K_i - q^{(i_1(i) \cdot \lambda)}).$$

The Verma module is defined as  $M(\lambda) \stackrel{\text{def}}{=} U/J_\lambda$  and generated by the coset of 1, denoted  $v_\lambda$ . This has the following universal property: if  $M$  is a  $U$ -module and  $v \in M^\lambda$  is a vector of weight  $\lambda$  such that  $E_i v = 0$  for all  $i \in I$ , then there exists a unique  $U$ -module homomorphism  $\varphi : M(\lambda) \rightarrow M$  with  $\varphi(v_\lambda) = v$ .

The Verma module  $M(\lambda)$  has a unique maximal submodule. The Weyl module  $L(\lambda)$  is defined to be the unique maximal simple factor of  $M(\lambda)$ .



## Chapter 3

# Lie induction

Lie bialgebras occur as the principal objects in the infinitesimalisation of the theory of quantum groups—the semi-classical theory. Their relationship with the quantum theory has made available some new tools that we can apply to classical questions. In this chapter, we study the simple complex Lie algebras using the double-bosonisation construction of Majid. This construction expresses algebraically the induction process given by adding and removing nodes in Dynkin diagrams, which we call Lie induction.

We first analyse the deletion of nodes, corresponding to the restriction of adjoint representations to subalgebras. This uses a natural grading associated to each choice of deletion. We give explicit calculations of the module and algebra structures in the case of the deletion of a single node from the Dynkin diagram for a simple Lie (bi-)algebra in the Appendix on page 119.

We next consider the inverse process, namely that of adding nodes, and give some necessary conditions for the simplicity of the induced algebra. Finally, we apply these to the exceptional series of simple Lie algebras, in the context of finding obstructions to the existence of finite-dimensional simple complex algebras of types  $E_9$ ,  $F_5$  and  $G_3$ . In particular, our methods give a new point of view on why there cannot exist such an algebra of type  $E_9$ .

Throughout, unless otherwise stated, we work over the field of complex numbers  $\mathbb{C}$ , although many of the definitions and some of the results can be extended to fields of characteristic not two. We will make further comment on this later.

### 3.1 Deletion

For an algebra-subalgebra pair  $(\mathfrak{g}, \mathfrak{g}_0)$ ,  $\mathfrak{g}_0 \subset \mathfrak{g}$ , with  $\mathfrak{g}$  and  $\mathfrak{g}_0$  semisimple, define the *corank* of  $\mathfrak{g}_0$  in  $\mathfrak{g}$  to be  $\text{corank}(\mathfrak{g}, \mathfrak{g}_0) = \text{rank}(\mathfrak{g}) - \text{rank}(\mathfrak{g}_0)$ . In this chapter, we will mostly concern ourselves with corank-one pairs. We will later comment on the extension to higher coranks.

Our aim is to associate to each corank-one pair of finite-dimensional semisimple complex Lie bialgebras  $(\mathfrak{g}, \mathfrak{g}_0)$ ,  $\mathfrak{g}_0 \subset \mathfrak{g}$ , a  $\mathfrak{g}_0$ -module  $\mathfrak{b} = \mathfrak{b}(\mathfrak{g}, \mathfrak{g}_0)$  which, with the additional structure of a braided-Lie bialgebra, realises the induction from  $\mathfrak{g}_0$  to  $\mathfrak{g}$  given by an isomorphism of Lie bialgebras of  $\mathfrak{g}$  with the double-bosonisation (Theorem 2.2.2)  $\mathfrak{b} \bowtie \hat{\mathfrak{g}}_0 \bowtie \mathfrak{b}^{*\text{op}}$ . Here,  $\hat{\mathfrak{g}}_0$  denotes a suitable central extension of  $\mathfrak{g}_0$  which raises the rank by one.

To do this, we use a combination of structure theory and representation theory to give some general tools, described below, and we give our explicit calculations in an Appendix. It is clear that without loss of generality we may assume the larger algebra  $\mathfrak{g}$  is simple. However, we do not in general assume that the subalgebra  $\mathfrak{g}_0$  is simple.

#### 3.1.1 Gradings associated to simple roots

We exhibit here a  $\mathbb{Z}$ -grading associated to each choice of simple root in a Lie algebra  $\mathfrak{g}$ . It is this grading that will give us most of the information we need to determine the braided-Lie bialgebra  $\mathfrak{b}$  discussed above.

Choose a Cartan subalgebra  $\mathfrak{h}$  for  $\mathfrak{g}$ , a simple complex Lie algebra, and let  $R$  be the associated root system. Let  $S = \{\alpha_1, \dots, \alpha_l\}$  be a base of simple positive roots for  $R$  where  $l = \dim \mathfrak{h} = \text{rank}(\mathfrak{g})$ . Choose a Weyl basis for  $\mathfrak{g}$ , as follows:  $\mathfrak{g}$  is generated by elements  $H_i \in \mathfrak{h}$  corresponding to the  $\alpha_i$  and elements  $X_i^+ \in \mathfrak{g}^{\alpha_i}$ ,  $X_i^- \in \mathfrak{g}^{-\alpha_i}$  satisfying  $[X_i^+, X_i^-] = H_i$ . In particular, we have the Weyl relations

$$\begin{aligned} [H_i, X_j^+] &= C_{ij} X_j^+, \\ [H_i, X_j^-] &= -C_{ij} X_j^-, \\ [X_i^+, X_j^-] &= 0 \quad \text{if } i \neq j \end{aligned}$$

where  $C$  is the Cartan matrix associated to  $\mathfrak{g}$ . The full basis is

$$\{H_i, X_i^+, X_i^- \mid 1 \leq i \leq l\} \cup \{X_\alpha^+, X_\alpha^- \mid X_\alpha^\pm \in \mathfrak{g}^{\pm\alpha}, \alpha \in R^+ \setminus S\},$$

where  $R^+$  is the set of positive roots in  $R$ . We could use the root datum setting described in Section 2.1.1 but we will see that all the information we will need is specified by knowledge of the elements of  $R^+$ .

We will want to consider subsets of the negative roots to define  $\mathfrak{b}$  and we will work with the coordinate system provided by  $S$ , i.e. if  $\alpha$  is a root we can write  $\alpha = \sum_{i=1}^l k_i \alpha_i$  and we have all  $k_i \geq 0$  if and only if  $\alpha \in R^+$  and conversely all  $k_i \leq 0$  if and only if  $\alpha \in R^-$ , the set of negative roots. When  $\alpha$  is written in this way, we may define the support of  $\alpha$  to be the set  $\text{Supp}(\alpha) = \{\alpha_i \in S \mid k_i \neq 0\}$ . We will call  $k_i$  the multiplicity of  $\alpha_i$  in  $\alpha$ :  $\text{mult}_i(\alpha) \stackrel{\text{def}}{=} k_i$ . Finally, define the height of  $\alpha$  to be  $\text{ht}(\alpha) = \sum_{i=1}^l k_i$ , if  $\alpha = \sum_{i=1}^l k_i \alpha_i$ . In particular, the simple roots  $\alpha_i \in S$  have height one.

Since we have assumed  $\mathfrak{g}$  to be simple, there exists a unique root  $\Lambda$  in  $R^+$  with maximal height, i.e.  $\text{ht}(\alpha) < \text{ht}(\Lambda)$  for all  $\alpha \neq \Lambda$ ,  $\alpha \in R$  (see, for example, [Hum78, Lemma 10.4.A]). We will call  $\Lambda$  the highest root in  $R$ . We recall that  $X_\Lambda^+$  is the highest weight vector in the adjoint representation. The coordinate expression for  $\Lambda$  as a root,  $\Lambda = \sum_{i=1}^l m_i \alpha_i$ , may therefore be obtained from the expression for  $\Lambda$  in the basis of fundamental weights via multiplication by  $C^{-1}$ . For later use, we record these dual expressions for the irreducible root systems (labelled by the Cartan type) in Table 3.1. In what follows, we use parentheses  $(., \dots, .)$  for vectors in the root basis provided by  $S$  and square brackets  $[., \dots, .]$  for weights using the fundamental weights  $\{\omega_i \mid 1 \leq i \leq l\}$  (dual to  $S$ ) as basis. We will use the notation  $\omega_0$  for the zero weight  $[0, \dots, 0]$ .

Observe also that  $\text{ht}(\Lambda) = h - 1$ , where  $h$  is the Coxeter number of  $\mathfrak{g}$  ([Bou68, Ch. 6, Prop. 1.11.31]).

Let  $J$  be a subset of  $\{1, \dots, l\}$ . A root deletion of  $J$  is defined to be a 4-tuple  $(\mathfrak{g}, J, \mathfrak{g}_0, \iota)$ , where  $\mathfrak{g}_0$  is a subalgebra of  $\mathfrak{g}$  generated by a choice of  $3(l - |J|)$  generators  $\{\bar{H}_i, \bar{X}_i^+, \bar{X}_i^- \mid i \notin J\}$  and  $\iota$  is the Lie algebra map extending a choice of set map  $\bar{\iota}: \{\bar{H}_i, \bar{X}_i^+, \bar{X}_i^- \mid i \notin J\} \rightarrow \{H_i, X_i^+, X_i^- \mid i \notin J\}$  to a map  $\mathfrak{g}_0 \hookrightarrow \mathfrak{g}$ . Note that to give  $\bar{\iota}$  and hence  $\iota$ , it is sufficient to give a permutation on the complement  $\bar{J}$  of  $J$  in  $\{1, \dots, l\}$ .

Type	Highest root, $\Lambda$	$\text{ht}(\Lambda)$	$\omega_{\text{ad}}$ , highest weight of adjoint representation
$A_l$	$(1, 1, \dots, 1)$	$l$	$[1, 0, 0, \dots, 0, 1] = \omega_1 + \omega_l$
$B_l$	$(1, 2, 2, \dots, 2)$	$2l - 1$	$[0, 1, 0, \dots, 0] = \omega_2$
$C_l$	$(2, 2, \dots, 2, 1)$	$2l - 1$	$[2, 0, 0, \dots, 0] = 2\omega_1$
$D_l$	$(1, 2, 2, \dots, 2, 1, 1)$	$2l - 3$	$[0, 1, 0, \dots, 0] = \omega_2$
$E_6$	$(1, 2, 2, 3, 2, 1)$	11	$[0, 1, 0, 0, 0, 0] = \omega_2$
$E_7$	$(2, 2, 3, 4, 3, 2, 1)$	17	$[1, 0, 0, 0, 0, 0, 0] = \omega_1$
$E_8$	$(2, 3, 4, 6, 5, 4, 3, 2)$	29	$[0, 0, 0, 0, 0, 0, 0, 1] = \omega_8$
$F_4$	$(2, 3, 4, 2)$	11	$[1, 0, 0, 0] = \omega_1$
$G_2$	$(3, 2)$	5	$[0, 1] = \omega_2$

Table 3.1: Expressions for highest roots in the irreducible root systems

In the case when  $|J| = 1$ ,  $J = \{\alpha_d\}$  we write  $(\mathfrak{g}, d, \mathfrak{g}_0, \iota)$ . Clearly, the Dynkin diagram for  $\mathfrak{g}_0$  is given by deleting the nodes in the Dynkin diagram for  $\mathfrak{g}$  corresponding to the  $\alpha_j$ ,  $j \in J$ . The map  $\iota$  defines an embedding of the Dynkin diagram of  $\mathfrak{g}_0$  into that for  $\mathfrak{g}$  in the obvious way.

We will use the following shorthand notations for  $\iota$ . We write  $\text{id}$  if  $\iota$  is the identity map or write  $\iota$  ‘algebraically’, if possible. For example, we may write  $i \mapsto i + 1$  for the embedding of (the diagram)  $A_3$  into  $A_4$  given by  $1 \mapsto 2$ ,  $2 \mapsto 3$ ,  $3 \mapsto 4$ . Otherwise we will write  $\iota$  in two-row permutation notation, with  $\mathfrak{g}_0$  on top, although it will not be a genuine permutation as the label sets will differ.

We now restrict to the case  $|J| = 1$ ,  $J = \{\alpha_d\}$ , i.e. the deletion of one simple root. Let  $\mathfrak{g}$  be a finite-dimensional complex simple Lie algebra.

**Lemma 3.1.1.** *Associated to each simple root  $\alpha_d \in S$ , there is a  $\mathbb{Z}$ -grading of  $\mathfrak{g}$  given by the  $\alpha_d$ -multiplicity as follows. Define  $\text{mult}_d(X_\alpha^\pm) = \text{mult}_d(\alpha)$ ,  $\alpha \in R$ , and  $\text{mult}_d(H_i) = 0$  for all  $i = 1, \dots, l$ . Set*

$$\mathfrak{g}_{[i]} = \text{span}_{\mathbb{C}}\{x \in \mathfrak{g} \mid \text{mult}_d(x) = i\},$$

with the convention  $\text{span}_{\mathbb{C}}\{\emptyset\} = \{0\}$ . Then  $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{[i]}$ .

*Proof:* This is immediate from the additivity of  $\text{mult}_d(-)$ , coming from the additivity in the root system.  $\square$

Note that this is not the trivial  $\mathbb{Z}$ -grading of a finite-dimensional simple Lie algebra, with  $\mathfrak{g}$  as the zero part and all other components zero. In the above grading, the zero part is  $\mathfrak{g}_{[0]} = \tilde{\mathfrak{g}}_0$ , a central extension of the subalgebra  $\mathfrak{g}_0 \subset \mathfrak{g}$  generated by all the generators of  $\mathfrak{g}$  except  $H_d$ ,  $X_d^+$  and  $X_d^-$ . The number of non-zero graded components is  $2 \cdot \text{mult}_d(\Lambda) + 1$  ( $\Lambda$  the highest root in  $\mathfrak{g}$ ) and we see from Table 3.1 that we have  $1 \leq \text{mult}_d(\Lambda) = m_d \leq 6$  in general and  $m_d \leq 3$  if  $d$  is chosen such that  $\mathfrak{g}_0$  is simple.

The most important property of this grading is that it gives the restriction or branching of the adjoint representation of  $\mathfrak{g}$  to the subalgebra  $\mathfrak{g}_0$ .

**Proposition 3.1.2.** *For  $i \neq 0$ ,  $\mathfrak{g}_{[i]}$  is an irreducible  $\mathfrak{g}_0$ -module. Also,  $\mathfrak{g}_{[0]} = \mathfrak{g}_0 \oplus \mathbb{C}$  as  $\mathfrak{g}_0$ -modules.*

*Proof:* The action of  $\mathfrak{g}_0$  is induced by the bracket in  $\mathfrak{g}$  and it is then clear that the  $\mathfrak{g}_{[i]}$  are  $\mathfrak{g}_0$ -modules by the grading property.

That the  $\mathfrak{g}_{[i]}$ ,  $i \neq 0$ , are irreducible may be deduced from a result of Azad, Barry and Seitz ([ABS90]). Their results concern algebraic groups over more general fields but the parts we need are root system-theoretic and so carry across immediately. The appropriate theorem in their paper is Theorem 2.  $\square$

*Note:* We observe that for  $i = \pm 1, \pm m_d$ , the irreducibility of  $\mathfrak{g}_{[i]}$  is immediate. The modules  $\mathfrak{g}_{[\pm 1]}$  have a primitive generator, namely  $X_d^\pm$ ; for  $\mathfrak{g}_{[-1]}$ , this highest weight vector has highest weight given by the negative of the  $d$ th row of the Cartan matrix for  $\mathfrak{g}$  with the  $d$ th column deleted and re-ordered according to that induced by the embedding  $\iota : \mathfrak{g}_0 \hookrightarrow \mathfrak{g}$ . The modules  $\mathfrak{g}_{[\pm m_d]}$  have a unique lowest weight vector,  $X_\Lambda^\pm$ . These observations are useful for the calculations we perform later.

The above grading is related to double-bosonisation as follows. Let  $\mathfrak{n}^-$  be the standard negative Borel subalgebra of  $\mathfrak{g}$ , so  $\mathfrak{n}^- = \mathfrak{h} \oplus \sum_{\alpha \in R^-} \mathfrak{g}^\alpha$ . Let  $\mathfrak{b}$  be the Lie ideal of  $\mathfrak{n}^-$  generated by  $X_d^-$ . A basis for  $\mathfrak{b}$  is  $\{X_\alpha^- \mid \alpha_d \in \text{Supp}(\alpha)\}$  and we have the following.

**Proposition 3.1.3.** *Let  $\mathfrak{g}$  be a finite-dimensional simple complex Lie bialgebra. Choose a simple root of  $\mathfrak{g}$ ,  $\alpha_d$ . Then we have the decomposition*

$$\mathfrak{b} \bowtie \tilde{\mathfrak{g}}_0 \ltimes \mathfrak{b}^{*\text{op}}$$

with  $\mathfrak{g}_0$  generated by all the generators of  $\mathfrak{g}$  except  $H_d$ ,  $X_d^+$  and  $X_d^-$  and  $\mathfrak{b} = \bigoplus_{i < 0} \mathfrak{g}_{[i]}$ , a  $\mathbb{Z}$ -graded braided-Lie bialgebra.

*Proof:* This follows from Proposition 4.5 of [Maj00] and the definition of the grading associated to  $\alpha_d$  in Lemma 3.1.1.  $\square$

### 3.1.2 Automorphisms

Note that in both deletion and induction, we need to take account of the existence of graph automorphisms of some of the Dynkin diagrams associated to the simple Lie algebras, which we will call diagram automorphisms. In deletion, we see certain symmetries appearing in the results of our calculations. There are relatively few automorphisms to take care of—the list of simple Lie algebras with non-trivial automorphism group is as follows:  $A_l$  (with automorphism group  $\text{Aut } \mathfrak{g} = \mathbb{Z}/(2)$ ),  $D_4$  ( $S_3$ ),  $D_l$ ,  $l \geq 5$  ( $\mathbb{Z}/(2)$ ) and  $E_6$  ( $\mathbb{Z}/(2)$ ). Observe that these are all simply-laced algebras, that is, there is only one root length in the root system.

As a result of the existence of these automorphisms, we want to consider certain deletions (as defined in Section 3.1.1) equivalent. It is diagram automorphisms that lead us to insist on specifying the embedding  $\iota$  as part of the deletion data but we now record which give essentially the same data. By “essentially”, we mean that we may not find the same modules but may find their duals (where they are different). There will be  $|\text{Aut } \mathfrak{g}| \cdot |\text{Aut } \mathfrak{g}_0|$  equivalent deletions  $(\mathfrak{g}, -, \mathfrak{g}_0, -)$ . We list the equivalent deletions in Table 3.2.

We note that the potentially interesting case of the triple symmetry in the diagram for  $D_4$  does *not* yield three different representations but  $\mathfrak{g}_{[-1]} = V(\omega_2; A_3) = \bigwedge^2(V)$  in all cases.

$\mathfrak{g}$	$\mathfrak{g}_0$	Equivalent deletions
$A_{l+1}$	$A_l$	$(A_{l+1}, 1, A_l, i \mapsto i+1), (A_{l+1}, 1, A_l, i \mapsto l-i+2),$ $(A_{l+1}, l, A_l, \text{id}), (A_{l+1}, l, A_l, i \mapsto l-i+1)$
$D_4$	$A_3$	$(D_4, 1, A_3, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \end{pmatrix}), (D_4, 1, A_3, \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 3 \end{pmatrix}),$ $(D_4, 3, A_3, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}), (D_4, 3, A_3, \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix}),$ $(D_4, 4, A_3, \text{id}), (D_4, 4, A_3, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix})$
$D_{l+1}$	$A_l$	$(D_{l+1}, l-1, A_l, \begin{pmatrix} 1 & 2 & 3 & \dots & l-1 & l \\ 1 & 2 & 3 & \dots & l-1 & l+1 \end{pmatrix}),$ $(D_{l+1}, l-1, A_l, \begin{pmatrix} 1 & 2 & 3 & \dots & l-1 & l \\ l+1 & l-1 & l-2 & \dots & 2 & 1 \end{pmatrix}),$ $(D_{l+1}, l, A_l, \text{id}), (D_{l+1}, l, A_l, i \mapsto l-i+1)$
$E_6$	$D_5$	$(E_6, 1, D_5, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 5 & 4 & 3 & 2 \end{pmatrix}), (E_6, 1, D_5, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 5 & 4 & 2 & 3 \end{pmatrix}),$ $(E_6, 6, D_5, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix}), (E_6, 6, D_5, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix})$

Table 3.2: Equivalences of deletion data arising from diagram automorphisms.

### 3.1.3 Summary of deletions

We give here a summary of our calculations, with the details reserved for the Appendix. Firstly, in the above we did not consider how we obtained  $A_1 = \mathfrak{sl}_2$ , since  $A_1$  does not have a simple semisimple subalgebra. However, Majid observed in [Maj00] that the procedure of deleting all the roots from a Lie algebra, leaving just the Cartan subalgebra, and its corresponding induction make sense and he gives general formulæ there. *A priori*, this does not fit within our deletion scheme. However, we may consider  $\mathfrak{h} = \mathbb{C} \cdot H$  as the central extension of the zero Lie algebra (which has empty Dynkin diagram). Note that conventionally the zero Lie algebra is not considered to be simple, being Abelian. From the point of view of Lie induction, one could argue that it ought to be included in the classification.

We give the details of this deletion here:

$(A_1\mathbb{C})$  Deletion  $(A_1, 1, \mathfrak{h} = \mathbb{C} \cdot H, -)$

$\mathfrak{b}_{-1}$  is spanned by  $X^-$  and we have as action  $H \triangleright X^- = -2X^-$ . For the dual,  $\mathfrak{b}_1$ , we choose as basis  $X^+$  with  $X^+(X^-) = -1$  (the negative of the usual choice).

Then  $H \triangleright X^+ = 2X^+$ . We consider  $\mathfrak{b}_{-1} = \mathbb{C} \cdot X^-$  as a braided-Lie bialgebra with the zero bracket and cobracket and this induces the same for  $\mathfrak{b}_1$ . Note that the infinitesimal braiding is also zero.

One may check that  $\mathbb{C} \cdot H$  with the zero bracket, the quasitriangular structure  $r = \frac{1}{4}H \otimes H$  and the above action gives the double-bosonisation

$$\mathbb{C} \cdot X^- \bowtie \mathbb{C} \cdot H \bowtie \mathbb{C} \cdot X^+ \cong \mathfrak{sl}_2 = A_1$$

with the Drinfel'd–Sklyanin quasitriangular structure. Here we do not need to make a further central extension.

We now give a table (Table 3.3) summarising the remainder of our calculations, that is, for the (equivalence classes of) deletions  $(\mathfrak{g}, d, \mathfrak{g}_0, \iota)$  with  $\text{corank}(\mathfrak{g}, \mathfrak{g}_0) = 1$  and *both*  $\mathfrak{g}$  and  $\mathfrak{g}_0$  simple. From Proposition 3.1.3, we know that  $\mathfrak{b}$ , the braided-Lie bialgebra arising from deletion, is graded with irreducible components and it is these modules occurring in  $\mathfrak{b}$  that we give here. More details of the full braided-Lie bialgebra structure are given in the Appendix. We also indicate the type of representation, i.e. trivial, the natural representation, a spin representation, etc.



Table 3.3: Summary of deletions

$\mathfrak{g}$	$d$	$\mathfrak{g}_0$	$\iota$	$m_d$	$\mathfrak{b}_{-1} = \mathfrak{g}_{[-1]}$		$\mathfrak{b}_{-2} = \mathfrak{g}_{[-2]}$		$\mathfrak{b}_{-3} = \mathfrak{g}_{[-3]}$	
$A_{l+1}$	$l$	$A_l$	id	1	$\omega_l$	natural				
$B_{l+1}$	1	$B_l$	$i \mapsto i+1$	1	$\omega_1$	natural				
$C_{l+1}$	1	$C_l$	$i \mapsto i+1$	2	$\omega_1$	natural	$\omega_0$	trivial		
$D_{l+1}$	1	$D_l$	$i \mapsto i+1$	1	$\omega_1$	natural				
$E_7$	7	$E_6$	id	1	$\omega_6$					
$E_8$	8	$E_7$	id	2	$\omega_7$		$\omega_0$	trivial		
$B_{l+1}$	$l+1$	$A_l$	$i \mapsto l-i+1$	2	$\omega_1$	natural	$\omega_2$	$\bigwedge^2(\text{natural})$		
$C_{l+1}$	$l+1$	$A_l$	$i \mapsto l-i+1$	1	$2\omega_1$	$\text{Sym}^2(\text{natural})$				
$D_{l+1}$	$l+1$	$A_l$	$i \mapsto l-i+1$	1	$\omega_2$	$\bigwedge^2(\text{natural})$				
$E_6$	2	$A_5$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 6 \end{pmatrix}$	2	$\omega_3$	$\bigwedge^3(\text{natural})$	$\omega_0$	trivial		
$E_7$	2	$A_6$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$	2	$\omega_3$	$\bigwedge^3(\text{natural})$	$\omega_6$	$\bigwedge^6(\text{natural})$		
$E_8$	2	$A_7$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$	3	$\omega_3$	$\bigwedge^3(\text{natural})$	$\omega_6$	$\bigwedge^6(\text{natural})$	$\omega_1$	natural
$G_2$	1	$A_1$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	3	$\omega_1$	natural	$\omega_0$	trivial	$\omega_1$	natural
$G_2$	2	$A_1$	id	2	$3\omega_1$	$\text{Sym}^3(\text{natural})$	$\omega_0$	trivial		
$F_4$	1	$C_3$	$i \mapsto 5-i$	2	$\omega_3$		$\omega_0$	trivial		
$F_4$	4	$B_3$	id	2	$\omega_3$	spin	$\omega_1$	natural		
$E_6$	1	$D_5$	$i \mapsto 7-i$	1	$\omega_4$	positive spin				
$E_7$	1	$D_6$	$i \mapsto 8-i$	2	$\omega_5$	negative spin	$\omega_0$	trivial		
$E_8$	1	$D_7$	$i \mapsto 9-i$	2	$\omega_6$	positive spin	$\omega_1$	natural		

## 3.2 Induction

We now begin the programme to analyse the classification of the simple Lie algebras using the representation-theoretic approach of Lie induction. This gives a somewhat different perspective to the usual method for classifying the simple Lie algebras using the geometry and combinatorics of the root systems.

Our first task is to see to what general principles we can extract from the above analysis of deletions to give necessary conditions for braided-Lie bialgebras to induce simple Lie algebras. More precisely, we wish to analyse the properties required by a braided-Lie bialgebra  $\mathfrak{b}$  in the module category of a finite-dimensional simple Lie algebra  $\mathfrak{g}_0$  so that the double-bosonisation  $\mathfrak{b} \bowtie \tilde{\mathfrak{g}}_0 \bowtie \mathfrak{b}^{*\text{op}}$  is again finite-dimensional and simple.

In particular, we would like to understand what the obstructions are that limit the classification to the known series and exceptionals. This could suggest whether or not relaxing certain axioms would alter the classification, e.g. using quasi-Lie algebras. To do this, we use a classification of irreducible modules satisfying two key necessary conditions. This determines the modules which may appear as irreducible components in  $\mathfrak{b}$ , which we know to be graded. Then if no such modules exist for a given simple  $\mathfrak{g}_0$ , there can be no induction.

### 3.2.1 Necessary conditions for Lie induction

The key idea is that we are considering modules which are potential subsets of roots in irreducible root systems and the following conditions come from this and the structure discussed in the previous section. Firstly, we recall (Lemma 3.1.1) that we can consider a simple Lie algebra  $\mathfrak{g}$  to be  $\mathbb{Z}$ -graded by choosing a simple root  $\alpha_d$  and grading by  $\text{mult}_d(X_\alpha^\pm) = \text{mult}_d(\alpha)$ ,  $\text{mult}_d(H_i) = 0$ . So we have the condition

- (1)  $\mathfrak{b}$  should be a finite-dimensional graded braided-Lie bialgebra.

That is,  $\mathfrak{b} = \bigoplus_{j=-m}^{-1} \mathfrak{b}_j$ ,  $[\mathfrak{b}_j, \mathfrak{b}_k] \subseteq \mathfrak{b}_{j+k}$  (possibly zero) with  $m < \infty$ . Next, the homogeneous parts should be irreducible:

(2)  $\mathfrak{b}_j$  should be irreducible for all  $j = -1, \dots, -m$ .

This comes from the theorems of [ABS90]. For conditions on the candidates for the  $\mathfrak{b}_j$ , we look to the underlying irreducible root system. Any root system of a Lie algebra has one-dimensional root spaces so we require

(3)  $\mathfrak{b}_j$  has all its weight spaces one-dimensional.

Also, there can be at most two root lengths and the roots of the same length and height must be conjugate under the Weyl group of  $\mathfrak{g}_0$  ([ABS90]) so

(4)  $\mathfrak{b}_j$  has at most two Weyl group orbits.

This can be rephrased in terms of dominant weights:

(4')  $\mathfrak{b}_j$  has at most two dominant weights.

We will see that the conditions (3) and (4) are very restrictive when combined. We say a module is *defining* if it satisfies (3) and (4) (equivalently, (3) and (4')).

We also need a property related to the grading on  $\mathfrak{b}$ . When discussing calculating deletions, we observe that  $\mathfrak{b}_{-2}$  must be a submodule of  $\Lambda^2(\mathfrak{b}_{-1})$ , by considering the module map  $[\ , \ ]|_{\mathfrak{b}_{-1}}$ , and similarly for higher indices. So, we require

(5)  $\mathfrak{b}_k$  occurs as a submodule of  $\mathfrak{b}_i \otimes \mathfrak{b}_j$  for all  $i, j$  such that  $i + j = k$ .

This follows from considering the bracket maps, which will be module maps, and Schur's lemma. If  $i = j = k/2$ , we require that  $\mathfrak{b}_k$  occurs as a submodule of  $\Lambda^2(\mathfrak{b}_i)$ .

We will classify the defining modules for the simple Lie algebras in the next section but we can immediately see that the trivial module satisfies conditions (3) and (4) and so is a candidate. However, the following theorem discounts this possibility.

**Theorem 3.2.1.** *Let  $\mathfrak{g}$  be a finite-dimensional simple quasitriangular complex Lie bi-algebra and  $\mathbb{C}$  be its trivial representation. Then*

$$\mathbb{C} \rtimes \tilde{\mathfrak{g}} \ltimes \mathbb{C}^{*\text{op}} \cong \mathfrak{g} \oplus \mathfrak{sl}_2(\mathbb{C})$$

as Lie bialgebras, where  $\mathfrak{sl}_2(\mathbb{C})$  has the Drinfel'd-Sklyanin Lie cobracket.

*Proof:* Let  $\mathbb{C}$  be spanned by  $x^-$  and its dual  $\mathbb{C}^*$  be spanned by  $x^+$ . As braided-Lie bialgebras,  $\mathbb{C}$  and  $\mathbb{C}^*$  are trivial: they have zero bracket and braided-cobracket, by antisymmetry. Note that we can therefore dispense with the “op” on  $\mathbb{C}^*$ . We fix the dual pairing as  $\langle x^-, x^+ \rangle = 1$ .

We have made a central extension to  $\mathfrak{g}$ : explicitly, let this be  $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C} \cdot h$ . The central extension acts on  $\mathbb{C}$  by  $h \triangleright x^- = x^-$  and this induces  $h \triangleright x^+ = -x^+$  on the dual. This centrally-extended algebra becomes a bialgebra with the quasitriangular structure  $\tilde{r} = r + h \otimes h$ .

We now make the double-bosonisation and examine the resulting brackets. Firstly,  $[\mathfrak{g}, x^-] = [\mathfrak{g}, x^+] = 0$  since  $\mathfrak{g}$  is acting trivially and  $h$  spans a central extension, that is,  $[\mathfrak{g}, h] = 0$ . To see that we have a copy of  $\mathfrak{sl}_2(\mathbb{C})$  from  $\mathbb{C} \cdot x^- \oplus \mathbb{C} \cdot h \oplus \mathbb{C} \cdot x^+$ , we must calculate the bracket between  $x^-$  and  $x^+$  as given by the double-bosonisation formulæ. We have

$$\begin{aligned} [x^-, x^+] &= (x^-)_{(1)} \langle x^+, (x^-)_{(2)} \rangle + (x^+)_{(1)} \langle (x^+)_{(2)}, x^- \rangle \\ &\quad + 2\tilde{r}_+^{(1)} \langle x^+, \tilde{r}_+^{(2)} \triangleright (x^-) \rangle \\ &= 0 + 0 + 2r_+^{(1)} \langle x^+, r_+^{(2)} \triangleright (x^-) \rangle + 2h \langle x^+, h \triangleright x^- \rangle \\ &= 0 + 2h \langle x^+, x^- \rangle \\ &= 2h. \end{aligned}$$

We can re-choose our basis vectors as  $H = -2h$ ,  $X^- = x^-$ ,  $X^+ = x^+$  to see that we indeed have  $\mathfrak{sl}_2(\mathbb{C})$  as a Lie algebra.

So, we have  $\mathbb{C} \bowtie \tilde{\mathfrak{g}} \bowtie \mathbb{C}^{*\text{op}} \cong \mathfrak{g} \oplus \mathfrak{sl}_2(\mathbb{C})$  as Lie algebras and it remains to check that we have a direct sum as Lie coalgebras. Double-bosonisation gives us the Lie cobracket on  $\mathbb{C}$  as follows.

$$\begin{aligned} \delta X^- &= \delta x^- = \underline{\delta} x^- + \tilde{r}^{(2)} \otimes \tilde{r}^{(1)} \triangleright x^- - \tilde{r}^{(1)} \triangleright x^- \otimes \tilde{r}^{(2)} \\ &= 0 + r^{(2)} \otimes r^{(1)} \triangleright x^- - r^{(1)} \triangleright x^- \otimes r^{(2)} \\ &\quad + h \otimes h \triangleright x^- - h \triangleright x^- \otimes h \end{aligned}$$

$$\begin{aligned}
&= 0 + h \otimes x^- - x^- \otimes h \\
&= \frac{1}{2}(X^- \otimes H - H \otimes X^-) = \frac{1}{2}(X^- \wedge H).
\end{aligned}$$

Similarly, we have  $\delta X^+ = \frac{1}{2}(X^+ \wedge H)$ . Equivalently, we see that the quasitriangular structure given by double-bosonisation (Proposition 2.2.3) is

$$r^{\text{new}} = \tilde{r} + \sum_a f^a \otimes e_a = r + h \otimes h + x^+ \otimes x^- = r + \frac{1}{4}H \otimes H + X^+ \otimes X^-$$

where  $\sum_a f^a \otimes e_a$  is a sum over  $\{e_a\}$  a basis for  $\mathfrak{b} = \mathbb{C}$  and  $\{f^a\}$  is a dual basis. This is the Drinfel'd–Sklyanin quasitriangular structure. Hence, we have a direct sum as bialgebras.  $\square$

We note that this result is independent of the choice of quasitriangular structure on  $\mathfrak{g}$ , since  $\mathfrak{g}$  acts trivially in any case.

With respect to the induction procedure, this excludes the choice  $\mathfrak{b}_{-1} = V(\omega_0) = \mathbb{C}$  for all simple Lie algebras. Note that choosing  $\mathfrak{b}_{-1} = \mathbb{C}$  fixes  $\mathfrak{b}_j = 0$  for all  $j \leq -2$  by property (5) above together with the anti-symmetry required by a (graded) Lie bracket. We have  $\bigwedge^2 \mathbb{C} = 0$  and  $\mathfrak{b}_{-2}$  is required to be a submodule of this, so is zero, and this forces all the remaining  $\mathfrak{b}_j$  to be zero. So, to our list we add the property

(6)  $\mathfrak{b}_{-1}$  is not trivial.

An additional result, which we will refer to later, was given by Majid:

**Lemma 3.2.2** (cf. [Maj00, Corollary 4.2]). *Let  $\mathfrak{g}$  be a quasitriangular finite-dimensional simple complex Lie algebra and let  $\mathfrak{b}$  be a finite-dimensional irreducible representation with  $\bigwedge^2 \mathfrak{b}$  isotypical. Then the double-bosonisation  $\mathfrak{b} \rtimes \tilde{\mathfrak{g}} \ltimes \mathfrak{b}^{*\text{op}}$  is again simple quasitriangular and of strictly greater rank. Here,  $\mathfrak{b}$  has zero bracket and braided-cobracket and  $\tilde{\mathfrak{g}}$  is a (one-dimensional) central extension of  $\mathfrak{g}$ .*  $\square$

Unfortunately, as we will see in our explicit calculations, the above condition on  $\mathfrak{b}$  is rarely, if ever, satisfied outside the  $A$  series and so the usefulness of this lemma is limited. It remains, however, one of very few positive results (i.e. guaranteeing simplicity of the output) based on conditions on  $\mathfrak{b}$ .

### 3.2.2 Classification of defining modules

We now classify the irreducible defining modules for the simple Lie algebras, that is, those irreducible modules satisfying conditions (3) and (4) above. To do this, we combine a result of Howe (as described by Stembridge ([Ste03])) with some analysis of dominant weights. Howe's result classifies weight-multiplicity-free highest weight modules, that is, those with all weight spaces associated to non-zero weights being one-dimensional. This is almost property (3) above. We then examine this relatively short list to determine the defining modules for each simple Lie algebra.

In our notation and the terminology of Stembridge, Howe's result is as follows:

**Theorem 3.2.3** ([How95]). *Let  $\mathfrak{g}$  be a finite-dimensional simple complex Lie algebra. Then a non-trivial irreducible  $\mathfrak{g}$ -module  $V(\lambda)$  has one-dimensional weight spaces if and only if*

- i)  $\lambda$  is minuscule,
- ii)  $\lambda$  is quasi-minuscule and  $\mathfrak{g}$  has only one short simple root,
- iii)  $\mathfrak{g} = C_3 = \mathfrak{sp}_6$  and  $\lambda = \omega_3$ , or
- iv)  $\mathfrak{g} = A_l = \mathfrak{sl}_{l+1}$  and  $\lambda = m\omega_1$  or  $\lambda = m\omega_l$  for some  $m \in \mathbb{N}$ .

A weight  $\lambda$  is called minuscule if  $\langle \lambda, \alpha \rangle \leq 1$  for all  $\alpha \in R$ . In [Hum78], a dominant minuscule weight is called minimal and an alternative characterisation is given, namely, if  $\mu$  is also dominant and  $\mu \prec \lambda$  then  $\mu = \lambda$ . Here  $\prec$  is the usual partial ordering on weights. In [PSV98], non-zero minuscule dominant weights are called microweight. We include the zero weight in the minuscule weights. Minuscule weights are also discussed in the exercises to Chapter 6 of [Bou68]. Note that non-zero minuscule weights do not exist for all Dynkin types.

A weight  $\lambda$  is called quasi-minuscule if  $\langle \lambda, \alpha \rangle \leq 2$  for all  $\alpha \in R$  and there exists a unique  $\alpha' \in R$  such that  $\langle \lambda, \alpha' \rangle = 2$ . For an irreducible root system, there is a unique dominant quasi-minuscule weight, namely the short dominant root. The modules  $V(\lambda)$ ,  $\lambda$  quasi-minuscule, are called short-root representations in [PSV98].

Table 2 in [PSV98] gives the following lists of (non-zero) minuscule and quasi-minuscule weights:

**Non-zero minuscule weights:**

$$\begin{array}{ll} A_l & \omega_i, 1 \leq i \leq l \\ B_l & \omega_l \\ C_l & \omega_1 \end{array} \quad \begin{array}{ll} D_l & \omega_1, \omega_{l-1}, \omega_l \\ E_6 & \omega_1, \omega_6 \\ E_7 & \omega_7 \end{array}$$

**Quasi-minuscule weights:**

$$\begin{array}{ll} A_l & [1, 0, 0, \dots, 0, 1] \text{ (adjoint)} \\ B_l & \omega_1 \\ C_l & \omega_2 \\ D_l & \omega_2 \\ E_6 & \omega_2 \end{array} \quad \begin{array}{ll} E_7 & \omega_1 \\ E_8 & \omega_8 \\ F_4 & \omega_4 \\ G_2 & \omega_1 \end{array}$$

The modules satisfying (3) and (4) (the defining modules) can therefore be calculated by striking out of the above classification all those with too many orbits.

**Theorem 3.2.4** ([Baz04]). *Let  $\mathfrak{g}$  be a finite-dimensional simple complex Lie algebra. The following is a list of all weights  $\lambda$  such that the highest weight  $\mathfrak{g}$ -module  $V(\lambda)$  satisfies properties (3) and (4).*

$$\begin{array}{ll} A_1 & \omega_0, \omega, 2\omega, 3\omega \\ A_l, l \geq 2 & \omega_0, \omega_i (1 \leq i \leq l), 2\omega_1, 2\omega_l \\ B_l & \omega_0, \omega_1, \omega_l \\ C_3 & \omega_0, \omega_1, \omega_3 \\ C_l, l \geq 4 & \omega_0, \omega_1 \\ D_l & \omega_0, \omega_1, \omega_{l-1}, \omega_l \end{array} \quad \begin{array}{ll} E_6 & \omega_0, \omega_1, \omega_6 \\ E_7 & \omega_0, \omega_7 \\ E_8 & \omega_0 \\ F_4 & \omega_0 \\ G_2 & \omega_0, \omega_1 \end{array}$$

*Proof:* The trivial module  $V(\omega_0)$  satisfies (3) and (4) for all types. It is well-known that the minuscule weights give rise to modules with exactly one Weyl group orbit ([Hum78], [PSV98]). Indeed, this is often given as essentially the definition.

Taking the types with only one short simple root excludes  $\omega_2$  for  $C_l$  of the quasi-minuscule weights since  $C_l$  has  $l - 1$  short simple roots and to avoid repetitions in the

labelling we have  $l \geq 3$ . Of the remaining quasi-minuscule weights, we exclude the algebra-weight pairs corresponding to adjoint representations, namely

$$(A_l \ (l \geq 2), [1, 0, 0, \dots, 0, 1]), (D_l, \omega_2), (E_6, \omega_2), (E_7, \omega_1) \text{ and } (E_8, \omega_8),$$

since in these cases the zero weight occurs with multiplicity  $l$ , the rank of  $\mathfrak{g}$ , which is greater than one.

For  $(F_4, \omega_4)$ , the zero weight has multiplicity two, so is excluded. We find that  $V(\omega_3)$  for  $C_3$  (the long root representation) has two Weyl group orbits; the zero weight does not occur. (We used LiE ([vL94]) to obtain this information.)

For  $(A_1, m\omega \ (= m\omega_1 = m\omega_l))$ , we have  $\omega$  already—it is minuscule—and if  $m \geq 4$ , it is easy to see that  $V(m\omega)$  has more than two orbits. We are left with  $m = 2$  or  $3$ . For  $m = 2$ , we have  $V(2\omega) = \text{Sym}^2(V)$  ( $V$  the natural representation) and this has two orbits, the zero weight orbit and one other. For  $m = 3$ ,  $V(3\omega) = \text{Sym}^3(V)$  does not contain the zero weight but does have exactly two orbits.

Finally, for  $(A_l \ (l \geq 2), m\omega_1)$ ,  $m = 1$  is covered by the minuscule case and if  $m \geq 3$  there are more than two orbits, as is easily seen. However,  $(A_l \ (l \geq 2), 2\omega_1)$  is kept:  $V(2\omega_1) = \text{Sym}^2(V)$  ( $V$  the natural representation) has exactly two orbits. Since  $m\omega_l$  is dual to  $m\omega_1$ , we are done.  $\square$

### 3.2.3 Induction for the exceptional series

In the remainder, we examine the question of extending the (known) exceptional series. In particular, we show how our method indicates the obstructions to there being a finite-dimensional simple  $E_9$ ,  $F_5$  or  $G_3$ . There are several different levels at which these obstructions may occur. The first is that there may be no appropriate choices of modules to feed into the induction, as a result of the classification of the previous section. The second is that, should suitable modules exist, these may not admit a braided-Lie bialgebra structure, to satisfy property (1) of Section 3.2.1. Thirdly, although we might be able to find suitable candidates, we might find that candidates obtained via different induction routes may not coincide.



Our general algorithm is as follows, suggested by the six properties we listed in Section 3.2.1. We should take a simple algebra  $\mathfrak{g}_0$  of rank  $l$  and examine the list of defining modules in Theorem 3.2.4 to find a candidate  $V(\lambda_1)$  for  $\mathfrak{b}_{-1}$ , the first graded part of the braided-Lie bialgebra  $\mathfrak{b}$  we need. By Theorem 3.2.1, we exclude the trivial module  $V(\omega_0)$  as a choice for  $\mathfrak{b}_{-1}$ . Next calculate  $\bigwedge^2(V(\lambda_1))$ : if this is zero or has no irreducible submodules which are defining modules (for  $\mathfrak{g}_0$ ), we stop here. Otherwise, such a submodule, together with the zero subspace, is a candidate  $V(\lambda_2)$  for  $\mathfrak{b}_{-2}$ . We then see if there are non-zero maps from  $\mathfrak{b}_{-1} \otimes \mathfrak{b}_{-2}$  into any defining module  $V(\lambda_3)$  for  $\mathfrak{g}_0$  satisfying the properties of a bracket, namely anti-symmetry and the (graded) Jacobi identity. If there is such a map, then  $V(\lambda_3)$  is a candidate for  $\mathfrak{b}_{-3}$ , and we repeat the process, considering maps from  $\mathfrak{b}_j \otimes \mathfrak{b}_k$  to defining modules to find candidates for  $\mathfrak{b}_{j+k}$ .

We now apply this algorithm to the appropriate simple algebras of rank 8, 4 and 2, to see in what respect the construction of  $E_9$ ,  $F_5$  and  $G_3$  fails.

### $E_9$

The first obvious line of attack is to consider induction from  $E_8$ . We may deal with this easily, as Theorems 3.2.1 and 3.2.4 show that there are no possible choices for  $\mathfrak{b}_{-1}$  and hence no inductions. In fact, this is a stronger statement than we need as by considering the deletion from  $E_9$ , whose Cartan matrix we know, we would require  $\mathfrak{b}_{-1} = V(\omega_8; E_8) = E_8$  (the adjoint representation). Clearly, we cannot have this as there is the eight-dimensional Cartan subalgebra, so the zero weight space is not one-dimensional. This is the first type of obstruction described above.

Of course, we could also look to induce from the other series. If we consider induction from  $D_8$ , then we will require  $\mathfrak{b}_{-1} = V(\omega_7; D_8)$ ; the embedding of  $D_8$  in  $E_9$  we choose is  $\iota : i \mapsto 10 - i$ , where the labelling of the Dynkin diagram for  $E_9$  follows the usual pattern for  $E_l$ ,  $l = 6, 7, 8$ . As desired,  $V(\omega_7; D_8)$  is a defining module for  $D_8$  but we have  $\bigwedge^2(V(\omega_7; D_8)) = V(\omega_2; D_8) \oplus V(\omega_6; D_8)$  and neither of these is defining. So we are forced to take  $\mathfrak{b}_j = 0$  for  $j \leq -2$  and our first candidate for the braided-Lie bialgebra inducing  $E_9$  is  $\mathfrak{b} = V(\omega_7; D_8)$ . This would yield a Lie algebra of dimension 377.

At this point, we need to know if  $V(\omega_7; D_8)$  admits an appropriate braided-Lie bialgebra structure. We must have the zero braided-Lie bialgebra structure, since the grading property implies that both  $V(\omega_7; D_8)$  and its dual should be Abelian Lie algebras and so by dualisation the cobrackets must be zero. But for a module to be a braided-Lie bialgebra with the zero structures, the infinitesimal braiding  $\psi$  must be zero. In the case where  $\bigwedge^2 \mathfrak{b}$  is isotypical, we have Lemma 3.2.2, the proof of which shows that we may kill the cocycle  $\psi$  with the central extension. Of course, this lemma does not apply here since  $\bigwedge^2 V(\omega_7; D_8)$  is not isotypical. This illustrates the second type of obstruction described above.

For the third type, we will examine the induction route through  $A_8$ . To give the correct Dynkin diagram, we must choose  $\mathfrak{b}_{-1} = V(\omega_3; A_8)$ ; the embedding is  $\iota = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}$ . Now  $\bigwedge^2 V(\omega_3; A_8) = V([0, 1, 0, 1, 0, 0, 0, 0]; A_8) \oplus V(\omega_6; A_8)$  and  $V(\omega_6; A_8)$  is a defining module for  $A_8$  so we have the choice of  $V(\omega_6; A_8)$  and the zero space for  $\mathfrak{b}_{-2}$ . Next, we have (all as  $A_8$ -modules)

$$\begin{aligned} V(\omega_3) \otimes V(\omega_6) &= V([0, 0, 1, 0, 0, 1, 0, 0]) \oplus V([0, 1, 0, 0, 0, 0, 1, 0]) \\ &\quad \oplus V([1, 0, 0, 0, 0, 0, 0, 1]) \oplus V(\omega_0) \end{aligned}$$

and the first three terms are not defining but the last is. (We excluded  $V(\omega_0)$  as a choice for  $\mathfrak{b}_{-1}$  in Theorem 3.2.1 but it is valid as a choice for other  $\mathfrak{b}_j$  and indeed it does occur.) Hence we have the choices  $\mathfrak{b}_{-3} = V(\omega_0; A_8)$  or  $\mathfrak{b}_{-3} = 0$ . For  $\mathfrak{b}_{-4}$ , we have  $V(\omega_3) \otimes V(\omega_0) = V(\omega_3)$  and

$$\bigwedge^2(V(\omega_6)) = V([0, 0, 0, 0, 1, 0, 1, 0]) \oplus V(\omega_3)$$

so we can choose either  $V(\omega_3; A_8)$  or zero for  $\mathfrak{b}_{-4}$ . Finally, for  $\mathfrak{b}_{-5}$  and higher parts, we will see the same pattern, namely

$$\mathfrak{b}_j = \begin{cases} V(\omega_3; A_8) & \text{if } j \equiv -1 \pmod{3} \\ V(\omega_6; A_8) & \text{if } j \equiv -2 \pmod{3} \\ V(\omega_0; A_8) & \text{if } j \equiv 0 \pmod{3}. \end{cases}$$

Observe that

$$\dim \left( V(\omega_3; A_8) \rtimes \widetilde{A}_8 \ltimes V(\omega_3; A_8)^* \right) = 249 \quad \text{and}$$

$$\dim \left( (V(\omega_3; A_8) \oplus V(\omega_6; A_8)) \rtimes \widetilde{A}_8 \ltimes (V(\omega_3; A_8) \oplus V(\omega_6; A_8))^* \right) = 417.$$

Recall that the proposed candidate for  $E_9$  found by induction from  $D_8$  had dimension 377. These are clearly inconsistent: we have no sensible candidate for  $E_9$  which agrees from both the  $A$  and  $D$  inductions.

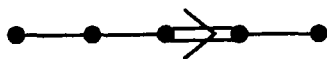
The above analysis suggests that it is very unlikely that in any enlarged scheme of finite-dimensional algebras we would find even a semisimple  $E_9$  candidate.

We have considered only Lie induction among finite-dimensional Lie algebras. Of course, we know of an (infinite-dimensional)  $E_9$ : the affine Kac-Moody algebra. Indeed, the general theory goes through to the extent that the Kac-Moody algebra  $E_9$  is obtainable by induction from  $E_8$ , via a (necessarily) infinite-dimensional braided-Lie bi-algebra. Further discussion of this would take us outside the problem we are considering, however.

In the above scenario, if we insist on the ‘flatness’ of the inductions across several routes, we would have to accept that no finite-dimensional  $E_9$  can exist as soon as we know that we cannot reach it from  $E_8$ . Then this instantly rules out any diagram with the diagram  $E_9$  as a sub-diagram: if there were a finite-dimensional  $E_{10}$ , say, then flatness would require that this had a deletion to a (finite-dimensional)  $E_9$ . Applying this principle, this reduces the number of cases to be considered in a proof of the classification of the simples significantly.

$F_5$

As for  $E_9$ , Theorems 3.2.1 and 3.2.4 exclude the possibility of an induction from the natural starting point  $F_4$ , giving rise to the candidate for  $F_5$  with Dynkin diagram



For the other choice, namely  $\mathfrak{b}_{-1} = V(\omega_4; F_4)$ , we simply use Theorem 3.2.4.

From the remaining series, we may start from either  $B_4$  or  $C_4$ . Similar arguments to the above rule out any induction from  $C_4$  but from  $B_4$ , we obtain the candidate  $V(\omega_4; B_4) \rtimes \widetilde{B}_4 \ltimes V(\omega_4; B_4)^*$  of dimension 69 corresponding to the diagram given above. As above, we must now consider whether or not the infinitesimal braiding is zero. The flatness question might appear not to arise, as we only have a single candidate, but since  $\dim F_4 = 52$ , for consistency we would need an  $F_4$ -module of dimension 8. Such a module does not exist.

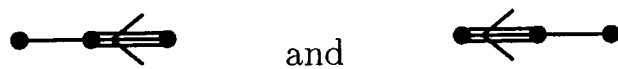
### $G_3$

For  $G_3$ , we may consider adding the new node to either the first node in  $G_2$  or the second. By Theorem 3.2.4, we cannot add it to the first, as this would require  $\mathfrak{b}_{-1} = V(\omega_2; G_2)$  and this is not a defining module for  $G_2$ . However,  $V(\omega_1; G_2)$  is a defining module so we may choose  $\mathfrak{b}_{-1} = V(\omega_1; G_2)$ . Then  $\bigwedge^2(V(\omega_1; G_2)) = V(\omega_1; G_2) \oplus V(\omega_2; G_2)$ , so we may choose  $\mathfrak{b}_{-2} = 0$  or  $\mathfrak{b}_{-2} = V(\omega_1; G_2)$ . If we choose the latter, we have an appropriate map to allow us to choose  $\mathfrak{b}_{-3} = V(\omega_1; G_2)$  and so on. Clearly, we cannot go on choosing  $V(\omega_1; G_2)$  forever so we must decide a suitable braided-Lie bialgebra structure exists for some  $m$  and, if it does, if

$$\left( \bigoplus_{j=-1}^{-m} V(\omega_1; G_2) \right) \rtimes \widetilde{G}_2 \ltimes \left( \bigoplus_{j=-1}^{-m} V(\omega_1; G_2) \right)^{\text{*op}} \quad (3.1)$$

is simple.

We may only obtain  $G_3$  from  $A_2$ , other than from  $G_2$ , but we may do this in two different ways (compare with the two embeddings of  $A_1$  in  $G_2$ ), leading to the possible diagrams



We may exclude the first of these by the usual appeal to Theorem 3.2.4:  $V(3\omega_i; A_2)$  is not defining for either  $i = 1$  or  $i = 2$ . For the second, we find ourselves in a similar

periodic situation to that for  $E_9$ , with

$$\mathfrak{b}_j = \begin{cases} V(\omega_1; A_2) & \text{if } j \equiv -1 \pmod{3} \\ V(\omega_2; A_2) & \text{if } j \equiv -2 \pmod{3} \\ V(\omega_0; A_2) & \text{if } j \equiv 0 \pmod{3}. \end{cases}$$

Some calculations with dimensions show that we do not have the same flatness problem for this induction and that discussed above from  $G_2$ . For example, choosing  $m = 2$  in (3.1) gives an algebra of dimension 43, which is consistent with choosing  $\mathfrak{b}_{-j}$  subject to the above rule and non-zero for  $j = 1, \dots, 7$ . Furthermore, since

$$\dim(V(\omega_1; A_2) \oplus V(\omega_2; A_2) \oplus V(\omega_0; A_2)) = 7,$$

we can find similar matching candidates for each choice of  $m$ . It may be that these modules do not admit the required braided-Lie bialgebra structures but we have given this example to illustrate that the flatness problem does not always arise.

We will discuss the three types of obstructions and potential means of overcoming them further in Chapter 6.

## Chapter 4

# Quantum Lie induction

We now turn our attention to quantum Lie induction, that is, the analysis of the ideas of the preceding chapter in the setting of quantized enveloping algebras. We begin by defining sub-root data, denoted  $\mathfrak{J} \subseteq_{\iota} \mathfrak{T}$ , which abstract the Lie algebra-subalgebra pairs we considered previously—or equivalently their Dynkin diagrams together with a suitable embedding. We then consider  $\mathbb{N}$ -graded Hopf algebras and give some general results on their structure, in particular that we obtain a bosonisation.

Such a grading of the quantum negative Borel subalgebra  $U_q^{\leq}(\mathfrak{T})$  may be associated to any choice of sub-root datum  $\mathfrak{J} \subseteq_{\iota} \mathfrak{T}$ . We analyse the structure of the zeroth homogeneous component of this grading, showing that it is a central extension of  $U_q^{\leq}(\mathfrak{J})$ , the quantum negative Borel subalgebra associated to  $\mathfrak{J}$ . This allows us to show that  $U_q(\mathfrak{T})$  may be expressed as a double-bosonisation of this central extension  $\widetilde{U_q(\mathfrak{J})}$  by an  $\mathbb{N}$ -graded Hopf algebra  $B = B(\mathfrak{T}, \mathfrak{J}, \iota)$  in the braided category of  $\widetilde{U_q(\mathfrak{J})}$ -modules.

We analyse the algebra structure of  $B$ , giving a set of generators. We also examine the module structure of  $B$  and see that  $B_1$  is a direct sum of (possibly quotients of) Weyl modules and that the higher graded components are sums of submodules of tensor products of these. Finally, we comment on the braided-coalgebra structure. In particular, the generators of  $B$  are braided-primitive.

### 4.1 Sub-root data

We define our principal object of study, a pair of suitably related root data.

**Definition 4.1.1.** *Let*

$$\mathfrak{T} = (I, \cdot, Y, X, <, >, i_1 : I \hookrightarrow Y, i_2 : I \hookrightarrow X)$$

$$\mathfrak{J} = (J, \cdot', Y', X', <', >', i'_1 : J \hookrightarrow Y', i'_2 : J \hookrightarrow X')$$

*be two root data. Then we say  $\mathfrak{J}$  is a sub-root datum of  $\mathfrak{T}$  via  $\iota$  if*

- i)  $\iota : J \hookrightarrow I$  is injective,*
- ii) the restriction of  $\cdot$  to the subgroup  $\mathbb{Z}[J] \subseteq \mathbb{Z}[I]$  (defined by the inclusion  $\iota(J) \subseteq I$ ) is  $\cdot'$ ,*
- iii)  $Y', X'$  are subgroups of  $Y, X$  respectively, such that  $Y/Y'$  and  $X/X'$  are free Abelian,*
- iv) the restriction of  $<, >$  to the subgroup  $Y' \times X' \subseteq Y \times X$  is  $<', >'$ ,*
- v)  $Y'$  and  $X'' \cong X/X'$  are orthogonal, i.e.  $\langle y', x'' \rangle = 0$  for all  $y' \in Y', x'' \in X''$  and*
- vi)  $i'_1 = i_1 \circ \iota$  and  $i'_2 = i_2 \circ \iota$ .*

*We will denote this by  $\mathfrak{J} \subseteq_{\iota} \mathfrak{T}$ .*

*Notes:*

- i) This definition makes use of the following result from Abelian group theory. If  $G/G'$  is a free Abelian quotient of an Abelian group  $G$ , then  $G = G' \oplus G''$  for some subgroup  $G''$  of  $G$  (see, for example, [Rob96, Section 4.2]). In particular,  $G'' \cong G/G'$ . Then  $v)$  makes sense, given  $iii)$ .*
- ii) The map  $\iota$  induces a Hopf algebra map  $U_q(\mathfrak{J}) \rightarrow U_q(\mathfrak{T})$ , which we will also denote by  $\iota$ . We will often identify  $U_q(\mathfrak{J})$  with its image under this Hopf algebra map.*

iii) We must specify the map  $\iota$ , rather than just a set inclusion  $J \subseteq I$ . For example, we distinguish between the following two embeddings of the subset  $J = \{1, \dots, l-1\}$  in  $I = \{1, \dots, l\}$ :

- $\iota_1 : J \rightarrow I, \iota_1(m) = m$
- $\iota_2 : J \rightarrow I, \iota_2(m) = l - m + 1$

We may also build up root data by taking direct sums.

**Definition 4.1.2.** *Let*

$$\mathfrak{T} = (I, \cdot, Y, X, <, >, i_1 : I \hookrightarrow Y, i_2 : I \hookrightarrow X)$$

$$\mathfrak{J} = (J, \cdot', Y', X', <', >', i'_1 : J \hookrightarrow Y', i'_2 : J \hookrightarrow X')$$

*be two root data. Then the direct sum  $\mathfrak{T} \oplus \mathfrak{J}$  of  $\mathfrak{T}$  and  $\mathfrak{J}$  is the root datum with*

- *underlying set  $I \cup J$ ,*
- *symmetric bilinear form  $\cdot_{\oplus} = \cdot \times \cdot'$  defined by*

$$i \cdot_{\oplus} j = \begin{cases} i \cdot j & \text{if } i, j \in I \\ i \cdot' j & \text{if } i, j \in J \\ 0 & \text{otherwise} \end{cases}$$

- *associated finitely generated free Abelian groups  $Y \oplus Y'$  and  $X \oplus X'$ ,*
- *non-degenerate bilinear form  $<, >_{\oplus} : (Y \oplus Y') \times (X \oplus X') \rightarrow \mathbb{Z}$  defined by*  

$$< y_1 \oplus y_2, x_1 \oplus x_2 >_{\oplus} = < y_1, x_1 > + < y_2, x_2 >'$$
- *associated inclusions  $i_j \oplus i'_j : I \cup J \rightarrow Y \oplus Y', j = 1, 2$ , with*  

$$(i_1 \oplus i'_1)|_I = i_1, \text{ etc.}$$

It is clear that this is again a root datum. The notions of sub-root datum and direct sum are suitably compatible:  $\mathfrak{T}, \mathfrak{J}$  are sub-root data of  $\mathfrak{T} \oplus \mathfrak{J}$  via the inclusions  $I, J \subseteq I \cup J$ . The quotients  $(Y \oplus Y')/(Y \oplus 0) \cong Y'$ ,  $(X \oplus X')/(0 \oplus X') \cong X$ , etc., are free Abelian,



as required. From the above formula for  $\langle \cdot, \cdot \rangle_\oplus$ , we see that  $Y$  (respectively  $Y'$ ) and  $X'$  (resp.  $X$ ) are orthogonal.

Let  $\mathfrak{J} \subseteq_\iota \mathfrak{T}$  be a sub-root datum of  $\mathfrak{T}$  via  $\iota$ .

**Definition 4.1.3.** We have the splitting  $X = X' \oplus X''$ , so let  $\pi : X \rightarrow X/X''$  be the canonical projection and  $i : X/X'' \rightarrow X'$  the isomorphism of  $X/X''$  with  $X'$ . Define the restriction map  $\rho : X \rightarrow X'$  to be  $\rho = i \circ \pi$ . In particular, we have  $\rho|_{X'} = \text{id}_{X'}$ . If  $\lambda \in X$ , we will often denote  $\rho(\lambda) \in X'$  by  $\lambda'$ . This is consistent with the decomposition  $\lambda = \lambda' \oplus \lambda''$ ,  $\lambda' \in X'$ ,  $\lambda'' \in X''$  given by  $X = X' \oplus X''$ .

The following lemma will be often used in what follows.

**Lemma 4.1.1.** For all  $\mu' \in Y'$ , we have  $\langle \mu', \rho(\lambda) \rangle' = \langle \mu', \lambda \rangle$ .

*Proof:* If  $\lambda = \lambda' \oplus \lambda''$ , we have  $\rho(\lambda) = \lambda'$  and

$$\begin{aligned} \langle \mu', \lambda \rangle &= \langle \mu', \lambda' \oplus \lambda'' \rangle \\ &= \langle \mu', \lambda' \rangle + \langle \mu', \lambda'' \rangle \\ &= \langle \mu', \lambda' \rangle + 0 && \text{since } Y' \text{ and } X'' \text{ are assumed orthogonal} \\ &= \langle \mu', \lambda' \rangle' && \text{since } \langle \cdot, \cdot \rangle, \langle \cdot, \cdot \rangle' \text{ agree on } Y' \times X' \\ &= \langle \mu', \rho(\lambda) \rangle'. \end{aligned} \quad \square$$

We call  $\rho$  the restriction map as it encodes the restriction of weight representations from  $U_q(\mathfrak{T})$  to  $U_q(\mathfrak{J})$ .

**Lemma 4.1.2.** Let  $\mathfrak{J} \subseteq_\iota \mathfrak{T}$  and let  $M = \bigoplus_{\lambda \in X} M_{\mathfrak{T}}^\lambda$  be a weight module for  $U_q(\mathfrak{T})$ . Then  $M$  is a weight module for  $U_q(\mathfrak{J})$  by restriction<sup>1</sup>, so we may write  $M = \bigoplus_{\lambda' \in X'} M_{\mathfrak{J}}^{\lambda'}$ . Furthermore,

$$M_{\mathfrak{J}}^{\lambda'} = \bigoplus_{\substack{\lambda \in X, \\ \rho(\lambda) = \lambda'}} M_{\mathfrak{T}}^\lambda = \bigoplus_{\lambda'' \in X''} M_{\mathfrak{T}}^{\lambda' + \lambda''}.$$

*Proof:* The exact sequence of Abelian groups

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X/X' \longrightarrow 0$$

---

<sup>1</sup>That is, the pull-back of the action of  $U_q(\mathfrak{T})$  on  $M$  via the (Hopf algebra) map  $\iota$ .

splits and hence the exact sequence of  $k$ -Hopf algebras

$$0 \longrightarrow k[X'] \longrightarrow k[X] \longrightarrow k[X]/k[X'] \longrightarrow 0$$

splits. (We may identify  $k[M]$  with  $k \otimes_{\mathbb{Z}} M$  for any monoid  $M$  and field  $k$ ; the functor  $k \otimes_{\mathbb{Z}} -$  is exact.) Let  $j : k[X] \rightarrow k[X']$  be the splitting map.

Now  $k[X']$  is a sub-coalgebra of  $k[X]$ , so is certainly a coideal of  $k[X]$ . Hence by Proposition 2.3.2, the coaction of  $k[X]$  on  $M$  defining the weight space grading pushes out along  $j$  to a coaction of  $k[X']$  on  $M$ , giving  $M$  a weight space grading for the action of  $U_q(\mathfrak{J})$  on  $M$ , given by restriction.

The remainder follows immediately.  $\square$

It is then natural to ask if  $\rho$  preserves dominance. We say a root datum with associated embedding  $i_1 : I \hookrightarrow Y$  is  $Y$ -regular if the set  $\text{Im } i_1$  is linearly independent in  $Y$ .

**Lemma 4.1.3.** *Let  $\mathfrak{T}$  be a  $Y$ -regular root datum and let  $\mathfrak{J} \subseteq_{\iota} \mathfrak{T}$ . Then if  $\lambda \in D_I^{\mathfrak{T}}(X)$ , we have  $\rho(\lambda) \in D_J^{\mathfrak{J}}(X')$ . Explicitly,*

$$\langle i_1(i), \lambda \rangle \in \mathbb{N} \ \forall i \in I \Rightarrow \langle i'_1(j), \rho(\lambda) \rangle' \in \mathbb{N} \ \forall j \in J.$$

*Proof:* Note that  $\mathfrak{T}$  being  $Y$ -regular implies  $\mathfrak{J}$  is also  $Y$ -regular so the sets of dominant weights are well-defined. We have  $\lambda = \rho(\lambda) + \lambda''$ ,  $\rho(\lambda) \in X'$ ,  $\lambda'' \in X''$ . So for  $j \in J$ ,

$$\begin{aligned} \langle i_1(\iota(j)), \lambda \rangle &= \langle i_1(\iota(j)), \rho(\lambda) + \lambda'' \rangle \\ &= \langle i_1(\iota(j)), \rho(\lambda) \rangle + 0 && \text{since } Y' \text{ and } X'' \text{ are orthogonal} \\ &= \langle i'_1(j), \rho(\lambda) \rangle'. \end{aligned} \quad \square$$

## 4.2 Gradings and split projections

Let  $(M, +)$  be a commutative monoid, with identity element denoted 0, and let  $k[M]$  be the associated monoid algebra for a field  $k$ .

**Definition 4.2.1.** *An  $M$ -graded  $k$ -Hopf algebra  $H = \bigoplus_{m \in M} H_m$  is a  $k$ -Hopf algebra in the category of right  $k[M]$ -comodules,  $\mathcal{M}^{k[M]}$ .*

That is, all the structure maps defining the Hopf algebra structure on  $H$  are morphisms in the category. A morphism  $f : A \rightarrow B$  in the category  $\mathcal{M}^{k[M]}$  means  $f(A_m) \subseteq B_m$  for all  $m \in M$ . Explicitly, we have:

1.  $H \otimes H$  is an  $M$ -graded Hopf algebra with the induced grading

$$(H \otimes H)_c = \bigoplus_{a+b=c} H_a \otimes H_b, \quad (a, b, c \in M),$$

2.  $m : H \otimes H \rightarrow H$  has  $m(H_a \otimes H_b) \subseteq H_{a+b}$ ,
3.  $k$  is  $M$ -graded by  $k_0 = k$ ,  $k_m = 0$ ,  $m \neq 0$ ,
4.  $\eta : k \rightarrow H$  has  $\eta(k_0 = k) \subseteq H_0$ ,
5.  $\Delta : H \rightarrow H \otimes H$  has  $\Delta(H_c) \subseteq (H \otimes H)_c = \bigoplus_{a+b=c} H_a \otimes H_b$ ,
6.  $\varepsilon : H \rightarrow k$  has  $\varepsilon(H_0) \subseteq k_0 = k$ ,  $\varepsilon(H_m) = 0$  if  $m \neq 0$ ,
7.  $S : H \rightarrow H$  has  $S(H_m) \subseteq H_m$ .

The following proposition relates  $\mathbb{N}$ -gradings<sup>2</sup> to split projections.

**Proposition 4.2.1.** *Let  $H = \bigoplus_{n \in \mathbb{N}} H_n$  be an  $\mathbb{N}$ -graded  $k$ -Hopf algebra. Then  $H_0$  is a sub-Hopf algebra of  $H$ . Let  $\pi : H \rightarrow H_0$  be defined by*

$$\pi(H_i) = \begin{cases} \text{id}|_{H_0} & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

*Then  $\pi$  is a projection of  $\mathbb{N}$ -graded Hopf algebras, split by the inclusion  $\iota : H_0 \hookrightarrow H$ . By this, we mean that  $\pi, \iota$  are morphisms in  $\mathcal{M}^{k[\mathbb{N}]}$  and Hopf algebra maps, such that  $\pi$  is surjective,  $\iota$  is injective and  $\pi \circ \iota = \text{id}_{H_0}$  (the splitting condition).  $H_0$  is  $\mathbb{N}$ -graded in the obvious way:  $(H_0)_0 = H_0$ ,  $(H_0)_i = 0$  ( $i > 0$ ).*

*Proof:* Clearly  $\pi$  and  $\iota$  are morphisms in the category, by definition.  $H_0$  is a sub-Hopf algebra of  $H$  by 5. above:  $(H \otimes H)_0 = \bigoplus_{a+b=0} H_a \otimes H_b$  for  $a, b \geq 0$ , and therefore

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<sup>2</sup>In the literature (e.g. [Swe69], [Abe80]),  $\mathbb{N}$ -gradings are usually just called gradings. However, we will also want to work with  $\mathbb{Z}$ -gradings and other monoid gradings, so we will specify the monoid with which we are working, unless it is immediately obvious from the context.

$\Delta(H_0) \subseteq (H \otimes H)_0 = H_0 \otimes H_0$  as required. Hence  $\iota$  is a Hopf algebra map. The map  $\pi$  is a Hopf algebra map: that it is an algebra map is clear and

- i) for  $c \neq 0$ ,  $(\pi \otimes \pi)(\Delta(H_c)) = (\pi \otimes \pi)(\bigoplus_{a+b=c} H_a \otimes H_b) = 0$  since one of  $a$  and  $b$  is non-zero. Also,  $\pi(H_c) = 0$  so  $\Delta_0(\pi(H_c)) = 0 = (\pi \otimes \pi)(\Delta(H_c))$  so  $\pi$  is a coalgebra map on  $H_c$ ,  $c \neq 0$ .
- ii) since  $H_0$  is a sub-Hopf algebra,  $\Delta(H_0) \subseteq H_0 \otimes H_0$  and  $\pi|_{H_0} = \text{id}$  show that  $\pi$  is a coalgebra map on  $H_0$ .

Here  $\Delta$  is the coproduct on  $H$  and  $\Delta_0 = \Delta|_{H_0}$ . It is easy to see that  $\pi$  commutes with the (graded) antipode  $S$ , so  $\pi$  is a Hopf algebra map. Finally,  $\pi \circ \iota = \text{id}_{H_0}$  by definition.  $\square$

Following on from the previous proposition, we can invoke the Radford–Majid theorem:

**Theorem 4.2.2** (cf. [Rad85], [Maj93]). *Let  $(H, H')$  be a dual pair of Hopf algebras with  $H$  and  $H'$   $\mathbb{N}$ -graded. Assume  $H'$  has an invertible antipode. Let  $H \xrightleftharpoons[\iota]{\pi} H_0$  be a split Hopf algebra projection. Then there is a Hopf algebra  $B$  in the braided category of  $D(H_0, H'_0)$ -modules such that  $B \rtimes H_0 \cong H$ .*  $\square$

Here  $D$  denotes the Drinfel'd double, as described in Definition 2.3.5, where  $H_0$  and  $H'_0$  are dually paired and  $D(H_0, H'_0) = H_0 \bowtie H'^{\text{op}}_0$ . Note that the dual pairing of  $H$  and  $H'$  does descend to a dual pairing of the sub-Hopf algebras  $H_0$  and  $H'_0$  and the (invertible) antipode of  $H'$  restricts to an invertible antipode on  $H'_0$ .

We have the following explicit descriptions of  $B$  and the isomorphism, from [Maj93]:

- i)  $B \stackrel{\text{def}}{=} \{b \in H \mid b_{(1)} \otimes \pi(b_{(2)}) = b \otimes 1\}$ .  $B$  is a subalgebra of  $H$ , namely the subalgebra of coinvariants of  $H$  under the coaction given by  $\beta(h) = h_{(1)} \otimes \pi(h_{(2)})$ .
- ii) We may alternatively describe  $B$  as the image of the surjective map  $\Pi : H \rightarrow H$  given by  $\Pi(h) = h_{(1)}((S \circ \iota \circ \pi)(h_{(2)}))$  for all  $h \in H$ . For  $b \in B$ ,

$$\Pi(b) = b_{(1)} S \circ \iota \circ \pi(b_{(2)}) = b S \circ \iota(1) = b,$$

i.e.  $\Pi|_B = \text{id}_B$ . We also note that  $\Pi$  is graded:  $\Pi(H_n) \subseteq H_n$  for all  $n \in \mathbb{N}$ , since  $\Pi$  is given by a composition of graded maps.

iii) The action of  $D(H_0, H'_0)$  on  $B$  is given as follows. Let  $b \in B$ . Then

- for  $h \in H_0$ ,  $h \triangleright b = \iota(h_{(1)})b(S \circ \iota)(h_{(2)}) = \text{Ad}_{\iota(h)}(b)$  (since  $\iota$  is a Hopf algebra map) and
- for  $a \in H'_0$ ,  $b \triangleleft a = \langle \pi(b_{(1)}), a \rangle b_{(2)}$ .

iv) The braided structures on  $B$  are: for  $b, c \in B$ ,

- the braided-coproduct  $\underline{\Delta}b = \Pi(b_{(1)}) \otimes b_{(2)}$ ,
- the braided antipode  $\underline{S}b = ((\iota \circ \pi)(b_{(1)}))Sb_{(2)}$ , and
- the braiding  $\Psi(b \otimes c) = (\pi(b_{(1)}) \triangleright c) \otimes b_{(2)}$ .

v) The isomorphism  $\Upsilon : H \rightarrow B \rtimes H_0$  is given by

$$\Upsilon(h) = \Pi(h_{(1)}) \otimes \pi(h_{(2)}) = h_{(1)}((S \circ \iota \circ \pi)(h_{(2)})) \otimes \pi(h_{(3)})$$

for all  $h \in H$ . Its inverse is  $\Upsilon^{-1} : B \rtimes H_0 \rightarrow H$ ,

$$\Upsilon^{-1}(b \otimes h) = b \cdot \iota(h)$$

for  $b \in B$ ,  $h \in H_0$  and  $\cdot$  the product in  $H$ .

Recall that any Hopf algebra  $H$  acts on itself by  $\text{Ad}_u(v) = u_{(1)}vSu_{(2)}$  for  $u, v \in H$ . This is the adjoint action of  $H$  on itself.

**Lemma 4.2.3.**  $\text{Ad} : H \otimes H \rightarrow H$  is a graded map.

*Proof:* For  $u, v \in H$  of homogeneous degree,  $\deg(u) = \deg(u_{(1)}) + \deg(Su_{(2)})$  since  $\Delta$  and  $S$  are graded. Hence  $\deg(\text{Ad}_u(v)) = \deg(u) + \deg(v)$ .  $\square$

As we saw above,  $H_0$  acts on  $B$ . In fact,  $H$  acts on  $B$ .

**Lemma 4.2.4.**  $B$  is an  $\text{Ad}$ -submodule of  $H$ , that is, for all  $h \in H$ ,  $b \in B$ ,  $\text{Ad}_h(b) \in B$ .

*Proof:* By definition,  $((\text{id} \otimes \pi) \circ \Delta)(b) = b \otimes 1$ . Recalling that  $\pi$  is a Hopf algebra homomorphism, we have the following explicit calculation:

$$\begin{aligned}
 ((\text{id} \otimes \pi) \circ \Delta)(\text{Ad}_h(b)) &= ((\text{id} \otimes \pi) \circ \Delta)(h_{(1)}bSh_{(2)}) \\
 &= (\text{id} \otimes \pi)(h_{(1)}b_{(1)}Sh_{(4)} \otimes h_{(2)}b_{(2)}Sh_{(3)}) \\
 &= h_{(1)}b_{(1)}Sh_{(4)} \otimes \pi(h_{(2)}b_{(2)}Sh_{(3)}) \\
 &= h_{(1)}b_{(1)}Sh_{(4)} \otimes \pi(h_{(2)})\pi(b_{(2)})\pi(Sh_{(3)}) \\
 &= h_{(1)}bSh_{(4)} \otimes \pi(h_{(2)})\pi(Sh_{(3)}) \\
 &= h_{(1)}bSh_{(4)} \otimes \pi(h_{(2)}Sh_{(3)}) \\
 &= h_{(1)}bSh_{(3)} \otimes \pi(\varepsilon(h_{(2)})) \\
 &= h_{(1)}\varepsilon(h_{(2)})bSh_{(3)} \otimes 1 \\
 &= h_{(1)}bSh_{(2)} \otimes 1 \\
 &= \text{Ad}_h(b) \otimes 1
 \end{aligned}$$

and hence  $\text{Ad}_h(b) \in B$ . □

Next, we show that  $B$  inherits an  $\mathbb{N}$ -grading from  $H = \bigoplus_{n \in \mathbb{N}} H_n$ .

**Lemma 4.2.5.** *Let  $h \in H_n$  be homogeneous of degree  $n$ . Then  $\Upsilon(h) \in H_n \otimes H_0$  where  $\Upsilon : H \rightarrow H \otimes H$  is given by  $\Upsilon(h) = h_{(1)}((S \circ \iota \circ \pi)(h_{(2)})) \otimes \pi(h_{(3)})$ . That is,  $\Upsilon(H_n) \subseteq H_n \otimes H_0$ .*

*Proof:* First, recall that the maps  $m, \Delta, S, \iota$  and  $\pi$  are graded. We set  $\Delta^{p-1} : H \rightarrow H^{\otimes p}$ ,

$$\Delta^{p-1} \stackrel{\text{def}}{=} (\Delta \otimes \text{id}^{\otimes p-2}) \circ (\Delta \otimes \text{id}^{\otimes p-3}) \circ \cdots \circ (\Delta \otimes \text{id}) \circ \Delta.$$

(By the axioms for a coproduct,  $\Delta^{p-1}$  is independent of the arrangement of  $\text{id}$ 's and  $\Delta$  in each term.) Now  $H^{\otimes p}$  is  $\mathbb{N}$ -graded by

$$(H^{\otimes p})_n = \bigoplus_{c \models n} H_{c_1} \otimes \cdots \otimes H_{c_l}$$

where “ $\models$ ” denotes “is a composition of”<sup>3</sup>. With respect to this grading,  $\Delta^{p-1}$  is a graded map, since  $\Delta = \Delta^1 : H \rightarrow H^{\otimes 2}$  is graded.

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<sup>3</sup>A composition  $c \models n \in \mathbb{N}$  is an ordered tuple  $(c_1, \dots, c_l)$ ,  $c_i \in \mathbb{N}$ , such that  $\sum_{i=1}^l c_i = n$ .

Hence,  $\Upsilon = (m \otimes \text{id}) \circ (\text{id} \otimes (S \circ \iota \circ \pi) \otimes \pi) \circ \Delta^2$  is a graded map, so  $\Upsilon(h)$  is an element of  $(H^{\otimes 2})_n = \bigoplus_{a+b=n} H_a \otimes H_b$ . However,  $\text{Im } \pi = H_0$  so

$$((\text{id} \otimes \pi \otimes \pi) \circ \Delta^2)(h) \in \bigoplus_{a+0+0=n} H_a \otimes H_0 \otimes H_0 = H_n \otimes H_0 \otimes H_0.$$

Therefore,  $\Upsilon(h) \in H_n \otimes H_0$ . □

We may therefore make the following definition.

**Definition 4.2.2.** Let  $n \in \mathbb{N}$ . Define  $B_n = \{b \in B \mid \Upsilon(b) \in H_n \otimes H_0\}$  and if  $b \in B_n$ , set  $\deg(b) = n$ .

**Theorem 4.2.6.**  $B$  is an  $\mathbb{N}$ -graded algebra:  $B = \bigoplus_{n \in \mathbb{N}} B_n$ . Also, for all  $n \in \mathbb{N}$ ,  $B_n = B \cap H_n$ .

*Proof:* We first show that  $B_n = B \cap H_n$ . We have  $B_n \subseteq B$  by definition and  $B_n \subseteq H_n$ , since if  $b \in B_n$  then  $\Upsilon(b) \in H_n \otimes H_0$ . But then  $\Upsilon^{-1}(\Upsilon(b)) \in H_n$ , i.e.  $b \in H_n$ . So,  $B_n \subseteq B \cap H_n$ .

Conversely, if  $x \in B \cap H_n$  then  $\Upsilon(x) \in H_n \otimes H_0$  (since  $x \in H_0$ ) and  $x \in B$  so  $x \in B_n$ . Then  $B_n = B \cap H_n$ . In particular,  $B_n$  is a vector subspace of  $B$  and for  $m \neq n$ ,  $B_m \cap B_n = B \cap (H_m \cap H_n) = \{0\}$ , since  $H = \bigoplus_{n \in \mathbb{N}} H_n$  is direct. Hence,  $\bigoplus_{n \in \mathbb{N}} B_n \subseteq B$ .

For the converse, let  $b \in B \setminus \bigoplus_n B_n$ , for a contradiction. Since  $H$  is graded, we have a unique expression  $b = \sum_n h_n$  with  $h_n \in H_n$ . Then since  $b \in B$ ,

$$\begin{aligned} \sum_n (\text{id} \otimes \pi) \Delta(h_n) &= (\text{id} \otimes \pi) \Delta\left(\sum_n h_n\right) \\ &= (\text{id} \otimes \pi) \Delta(b) \\ &= b \otimes 1 \\ &= \left(\sum_n h_n\right) \otimes 1 \\ &= \sum_n (h_n \otimes 1). \end{aligned}$$

Now for  $h_n \in H_n$ ,  $(\text{id} \otimes \pi) \Delta(h_n) \in H_n \otimes H_0$ . Since the coproduct  $\Delta$  is graded,

$$\Delta(h_n) \in (H \otimes H)_n = \bigoplus_{a+b=n} H_a \otimes H_b = (H_n \otimes H_0) \oplus \bigoplus_{a+b=n, b \neq 0} H_a \otimes H_b.$$

By definition,  $\pi(H_b) = 0$  for  $b \neq 0$  so  $(\text{id} \otimes \pi)(\bigoplus_{a+b=n, b \neq 0} H_a \otimes H_b) = 0$  and hence  $(\text{id} \otimes \pi)\Delta(h_n) \in H_n \otimes H_0 \otimes H_0$ . Also,  $h_n \otimes 1 \in H_n \otimes H_0$  so  $\sum_n h_n \otimes 1$  and  $\sum_n (\text{id} \otimes \pi)\Delta(h_n)$  are two equal expressions in  $H \otimes H_0 = \bigoplus_n H_n \otimes H_0$  with  $h_n \otimes 1, (\text{id} \otimes \pi)\Delta(h_n) \in H_n \otimes H_0$  for all  $n$ . But  $H \otimes H_0$  is a *direct* sum and so by the uniqueness of expression in a direct sum we have  $h_n \otimes 1 = (\text{id} \otimes \pi)\Delta(h_n)$  for all  $n$ . Therefore,  $h_n \in B$ . So  $h_n \in B \cap H_n$  and  $b \in \bigoplus_n B_n$  : contradiction. Thus  $B = \bigoplus_n B_n$ .

Finally, we have the appropriate additivity: if  $b \in B_n, c \in B_m$  then the product  $bc$  lies in the intersection  $B \cap H_{n+m} = B_{n+m}$ , since  $B$  is a subalgebra and  $H$  is graded.  $\square$

Therefore we may focus our attention on the structure of the homogeneous components.

**Lemma 4.2.7.**  $B_n$  is a  $D(H_0, H'_0)$ -submodule of  $B$ .

*Proof:* As above,  $D(H_0, H'_0) = H_0 \bowtie H'^{\text{op}}_0$ . For  $h \in H_0, b \in B_n, h \triangleright b = \text{Ad}_{\iota(h)}(b)$ , which is graded, by Lemma 4.2.3:  $\deg(\text{Ad}_{\iota(h)}(b)) = \deg(\iota(h)) + \deg(b)$ . For  $a \in H'_0, b \in B_n, b \triangleleft a = \langle \pi(b_{(1)}), a \rangle b_{(2)}$ . Now  $\Delta$  is graded, so  $(\pi \otimes \text{id}) \circ \Delta(b) \in H_0 \otimes H_n$  and hence  $b \triangleleft a \in H_n$ , so  $b \triangleleft a \in B \cap H_n = B_n$ .  $\square$

**Lemma 4.2.8.** We have  $B_0 = k$ .

*Proof:* Firstly,  $k \subseteq H_0$  and if  $\lambda \in k$  then  $(\text{id} \otimes \pi)(\Delta(\lambda)) = \lambda(1 \otimes 1) = \lambda \otimes 1$  so  $\lambda \in B$  and  $k \subseteq B_0$ . For the converse, recall the definition of  $\Pi : H \rightarrow H$  above. For  $h \in H_0, \Pi(h) = h_{(1)}((S \circ \iota \circ \pi)(h_{(2)})) = h_{(1)}Sh_{(2)} = \varepsilon(h) \in k$ , i.e.  $\Pi(H_0) \subseteq k$ . So in particular,  $\Pi(B_0) \subseteq k$  but  $\Pi|_B = \text{id}_B$  so  $B_0 \subseteq k$ .  $\square$

### 4.3 The quantum negative Borel subalgebra $U_q^{\leq}(\mathfrak{T})$

Let  $\mathfrak{J} \subseteq_{\iota} \mathfrak{T}$ . Then  $U_q(\mathfrak{J})$  may be identified with the sub-Hopf algebra of  $U_q(\mathfrak{T})$  with generators  $E_j, F_j, j \in \iota(J), K_{\nu}, \nu \in Z' \stackrel{\text{def}}{=} \mathbb{Z}[i'_1(J)]$ .

Recall that  $U_q(\mathfrak{T})$  has a  $\mathbb{Z}[I]$ -grading, given by  $\deg E_i = -\deg F_i = i, \deg K_{\mu} = 0$ . However, it also has many  $\mathbb{Z}$ -gradings. Let  $\gamma : I \rightarrow \mathbb{Z}$  be any function. Then  $U_q(\mathfrak{T})$  is  $\mathbb{Z}$ -graded by  $\deg E_i = -\deg F_i = \gamma(i), \deg K_{\mu} = 0$ . We see this by noting that all the defining relations are homogeneous in degree (cf. [Kac90, Section 1.5]). In particular,



$U_q(\mathfrak{T})$  has a  $\mathbb{Z}$ -grading associated to any sub-root datum  $\mathfrak{J} \subseteq_{\iota} \mathfrak{T}$ . Let  $D = I \setminus \iota(J)$  and let  $\chi_D : I \rightarrow \{0, 1\}$  be the indicator function for  $D$ , i.e.

$$\chi_D(i) = \begin{cases} 1 & \text{if } i \in D \\ 0 & \text{if } i \notin D \end{cases}$$

Then, as above, regarding  $\chi_D$  as a function  $I \rightarrow \mathbb{Z}$  we have a  $\mathbb{Z}$ -grading on  $U_q(\mathfrak{T})$ :

$$U_q(\mathfrak{T}) = \bigoplus_{n \in \mathbb{Z}} U_q(\mathfrak{T})_{[n]}.$$

In particular,  $U_q(\mathfrak{J}) \subseteq U_q(\mathfrak{T})_{[0]}$  and  $\langle K_{\mu} \mid \mu \in Z \rangle \subseteq U_q(\mathfrak{T})_{[0]}$ .

Consider now the sub-Hopf algebra  $U_q^{\leq}(\mathfrak{T})$  of  $U_q(\mathfrak{T})$ , the analogue of the negative Borel subalgebra, generated by the set  $\{F_i \mid i \in I\} \cup \{K_{\mu} \mid \mu \in Z\}$ . Then  $U_q^{\leq}(\mathfrak{T})$  is  $\mathbb{N}$ -graded via  $\chi_D$ :  $\deg F_i = \chi_D(i)$ ,  $\deg K_{\mu} = 0$ . In particular,  $U_q^{\leq}(\mathfrak{T})_{[0]}$  contains  $U_q^{\leq}(\mathfrak{J})$ , which is generated by  $\{F_j \mid j \in \iota(J)\} \cup \{K_{\nu} \mid \nu \in Z'\}$ . Note, though, that  $U_q^{\leq}(\mathfrak{T}) \neq \bigoplus_{i \leq 0} U_q(\mathfrak{T})_{[i]}$  since for example  $E_i F_i \in U_q(\mathfrak{T})_{[0]}$  but  $E_i F_i \notin U_q^{\leq}(\mathfrak{T})$ . Also, as we recalled in Section 2.3.6,  $U_q^{\leq}(\mathfrak{T})$  is self-dually paired. Indeed  $(U_q^{\leq}(\mathfrak{T}), U_q^{\leq}(\mathfrak{T}))$  is a dual pair of  $\mathbb{N}$ -graded Hopf algebras. Hence, Proposition 4.2.1 and Theorem 4.2.2 apply to  $U_q^{\leq}(\mathfrak{T})$  and we have the following.

**Theorem 4.3.1.** *Let  $\mathfrak{J} \subseteq_{\iota} \mathfrak{T}$  be a sub-root datum of  $\mathfrak{T}$ . Then there exists a Hopf algebra  $B = B(\mathfrak{T}, \mathfrak{J}, \iota)$  in the braided category of  $D(U_q^{\leq}(\mathfrak{T})_{[0]})$ -modules such that*

$$U_q^{\leq}(\mathfrak{T}) \cong B \rtimes U_q^{\leq}(\mathfrak{T})_{[0]}.$$

□

Here we have

$$\begin{aligned} D(U_q^{\leq}(\mathfrak{T})_{[0]}) &= D(U_q^{\leq}(\mathfrak{T})_{[0]}, U_q^{\leq}(\mathfrak{T})_{[0]}) \\ &= U_q^{\leq}(\mathfrak{T})_{[0]} \bowtie (U_q^{\leq}(\mathfrak{T})_{[0]})^{\text{op}} \\ &= U_q^{\leq}(\mathfrak{T})_{[0]} \bowtie U_q^{\geq}(\mathfrak{T})_{[0]}. \end{aligned}$$

We now examine in more detail the structure of  $U_q^{\leq}(\mathfrak{T})_{[0]}$ .

### 4.3.1 The zeroth component $U_q^\leq(\mathfrak{T})_{[0]}$

We see immediately that the zeroth graded component  $U_q^\leq(\mathfrak{T})_{[0]}$  of  $U_q^\leq(\mathfrak{T})$  is generated by  $\{F_j \mid j \in \iota(J)\} \cup \{K_\mu \mid \mu \in Z\}$ . As noted above,  $U_q^\leq(\mathfrak{J}) \subseteq U_q^\leq(\mathfrak{T})_{[0]}$  and indeed is a sub-Hopf algebra. We show that  $U_q^\leq(\mathfrak{T})_{[0]}$  is a central extension of  $U_q^\leq(\mathfrak{J})$ . Observe that  $\mathbb{Z}[i_1'(J)] = Z' \subseteq Z = \mathbb{Z}[i_1(I)]$  and the quotient  $Z/Z'$  is free Abelian—the quotient may be identified with  $\mathbb{Z}[i_1(D)]$  where  $D = I \setminus J$ .

**Proposition 4.3.2.**  *$U_q^\leq(\mathfrak{T})_{[0]}$  is an extension (of Hopf algebras) of  $k[Z/Z']$  (the group Hopf algebra of  $Z/Z'$ ) by  $U_q^\leq(\mathfrak{J})$ . That is, we have the short exact sequence of bialgebras*

$$0 \longrightarrow U_q^\leq(\mathfrak{J}) \xhookrightarrow{i} U_q^\leq(\mathfrak{T})_{[0]} \xrightarrow{\pi} k[Z/Z'] \longrightarrow 0$$

and  $U_q^\leq(\mathfrak{T})_{[0]} \cong U_q^\leq(\mathfrak{J}) \otimes k[Z/Z']$  as a right  $k[Z/Z']$ -comodule and a left  $U_q^\leq(\mathfrak{J})$ -module.

*Proof:* Recall that if  $M$  is a monoid, the category of (right)  $k[M]$ -comodules is exactly that of  $M$ -graded vector spaces. So we want to identify a suitable  $k[Z/Z']$ -grading on  $H = U_q^\leq(\mathfrak{T})_{[0]}$  with  $H_{[0]} = U_q^\leq(\mathfrak{J})$ , so that we obtain  $U_q^\leq(\mathfrak{J})$  as the fixed point subalgebra under the coaction corresponding to this grading.

As observed above,  $Z/Z'$  is a free Abelian quotient of a free Abelian group  $Z$ , so we have  $Z = Z' \oplus Z''$  for some subgroup  $Z''$  of  $Z$  ( $Z''$  is isomorphic to  $\mathbb{Z}[i_1(D)]$ ). Explicitly, the sequence

$$0 \longrightarrow Z' \xrightarrow{i} Z \xrightarrow{\hat{\pi}} Z'' \longrightarrow 0$$

splits: we have  $\hat{j} : Z/Z' \rightarrow Z$  such that  $\hat{\pi} \circ \hat{j} = \text{id}_{Z/Z'}$  and  $Z'' = \text{Im } \hat{j} \cong Z/Z'$ . So, we have unique decomposition of elements of  $Z$  into elements of  $Z'$  and  $Z''$ : for  $\mu \in Z$ , we have  $\mu = \mu' \oplus \mu''$  for (unique)  $\mu' \in Z'$ ,  $\mu'' \in Z''$ . So to each  $\mu \in Z$  we have a unique associated pair  $(\mu', \nu)$  with  $\mu' \in Z'$ ,  $\nu = \hat{\pi}(\mu'') \in Z/Z'$ . Define  $p_1(\mu) = \mu'$ ,  $p_2(\mu) = \nu$  for all  $\mu \in Z$ .

Then the natural  $k[Z/Z']$ -grading is  $\deg F_i = 0$ ,  $\deg K_\mu = p_2(\mu)$ , on generators. We see that  $U_q^\leq(\mathfrak{J})$  is indeed the degree zero part of this grading. Let  $\{Q_\alpha \mid \alpha \in Z/Z'\}$  be a basis for  $k[Z/Z']$ . Hence we define the maps  $i : U_q^\leq(\mathfrak{J}) \hookrightarrow U_q^\leq(\mathfrak{T})_{[0]}$  to be inclusion and  $\pi : U_q^\leq(\mathfrak{T})_{[0]} \rightarrow k[Z/Z']$  by  $\pi(F_i) = Q_0$ ,  $i \in \iota(J)$ ,  $\pi(K_\mu) = Q_{p_2(\mu)}$  for all  $\mu \in Z$ , extended linearly and to products. We can express this as  $\pi(x) = Q_{\deg x}$  for all  $x \in U_q^\leq(\mathfrak{T})_{[0]}$ .

By definition,  $i$  and  $\pi$  are algebra maps and since  $U_q^\leq(\mathfrak{J})$  is a sub-Hopf algebra of  $U_q^\leq(\mathfrak{T})_{[0]}$ ,  $i$  is a bialgebra map. However,  $\Delta$  on  $U_q^\leq(\mathfrak{T})_{[0]}$  and  $\Delta_{k[Z/Z']}$  on  $k[Z/Z']$  are group-like on the generators  $K_\mu, Q_\nu$  so

$$(\pi \otimes \pi)(\Delta K_\mu) = Q_{p_2(\mu)} \otimes Q_{p_2(\mu)} = \Delta_{k[Z/Z']}(\pi(K_\mu)).$$

So  $\pi$  is a bialgebra map, also.

Now define a map  $f : U_q^\leq(\mathfrak{T})_{[0]} \rightarrow U_q^\leq(\mathfrak{J}) \otimes k[Z/Z']$  by  $f(F_i) = F_i \otimes Q_0$ ,  $i \in \iota(J)$ ,  $f(K_\mu) = K_{p_1(\mu)} \otimes Q_{p_2(\mu)}$  for all  $\mu \in Z$ , extended linearly and to products. It is clear that  $f$  is a bijection, with inverse  $f^{-1}(x \otimes Q_g) = xK_{\hat{j}(g)}$ ,  $\hat{j}$  as above. For all  $x \in U_q^\leq(\mathfrak{T})_{[0]}$ , we can express  $f(x)$  as  $f(x) = x' \otimes Q_{\deg x}$  for a unique  $x' \in U_q^\leq(\mathfrak{J})$ .

It remains to show that  $f$  is an isomorphism of right  $k[Z/Z']$ -comodules and left  $U_q^\leq(\mathfrak{J})$ -modules. The coaction of  $k[Z/Z']$  on  $U_q^\leq(\mathfrak{T})_{[0]}$  corresponding to the above grading is  $\beta : U_q^\leq(\mathfrak{T})_{[0]} \rightarrow U_q^\leq(\mathfrak{T})_{[0]} \otimes k[Z/Z']$ ,  $\beta(x) = x \otimes Q_{\deg x}$  —  $\beta$  exactly counts the degree. The left action of  $U_q^\leq(\mathfrak{J})$  on  $U_q^\leq(\mathfrak{T})_{[0]}$ ,  $\triangleright$ , is left multiplication. We must show that the following two diagrams commute:

$$\begin{array}{ccc} i) & U_q^\leq(\mathfrak{J}) \otimes U_q^\leq(\mathfrak{T})_{[0]} & \xrightarrow{\text{id} \otimes f} U_q^\leq(\mathfrak{J}) \otimes U_q^\leq(\mathfrak{J}) \otimes k[Z/Z'] \\ & \downarrow \triangleright & \downarrow m \otimes \text{id} \\ & U_q^\leq(\mathfrak{T})_{[0]} & \xrightarrow{f} U_q^\leq(\mathfrak{J}) \otimes k[Z/Z'] \end{array}$$

where  $m$  is multiplication in  $U_q^\leq(\mathfrak{J})$ , or equivalently, in  $U_q^\leq(\mathfrak{T})_{[0]}$ .

$$\begin{array}{ccc} ii) & U_q^\leq(\mathfrak{T})_{[0]} & \xrightarrow{f} U_q^\leq(\mathfrak{J}) \otimes k[Z/Z'] \\ & \downarrow \beta & \downarrow \text{id} \otimes \Delta_{k[Z/Z']} \\ & U_q^\leq(\mathfrak{T})_{[0]} \otimes k[Z/Z'] & \xrightarrow{f \otimes \text{id}} U_q^\leq(\mathfrak{J}) \otimes k[Z/Z'] \otimes k[Z/Z'] \end{array}$$

For  $i)$ , let  $x \in U_q^\leq(\mathfrak{J})$  and  $y \in U_q^\leq(\mathfrak{T})_{[0]}$ . Then

$$\begin{aligned} (f \circ \triangleright)(x \otimes y) &= f(xy) = f(x)f(y) \\ &= (x \otimes Q_0)(y' \otimes Q_{\deg y}) \\ &= xy' \otimes Q_{\deg y} \end{aligned}$$

and

$$\begin{aligned} ((m \otimes \text{id}) \circ (\text{id} \otimes f))(x \otimes y) &= (m \otimes \text{id})(x \otimes y' \otimes Q_{\deg y}) \\ &= xy' \otimes Q_{\deg y} \end{aligned}$$

as required.

For *ii*), let  $y \in U_q^\leq(\mathfrak{T})_{[0]}$ . Then

$$\begin{aligned} ((f \otimes \text{id}) \circ \beta)(y) &= (f \otimes \text{id})(y \otimes Q_{\deg y}) \\ &= y' \otimes Q_{\deg y} \otimes Q_{\deg y} \end{aligned}$$

and

$$\begin{aligned} ((\text{id} \otimes \Delta_{k[Z/Z']}) \circ f)(y) &= (\text{id} \otimes \Delta_{k[Z/Z']})(y' \otimes Q_{\deg y}) \\ &= y' \otimes Q_{\deg y} \otimes Q_{\deg y} \end{aligned}$$

as required.

So,  $f$  is a left  $U_q^\leq(\mathfrak{T})$ -module and right  $k[Z/Z']$ -comodule isomorphism and  $U_q^\leq(\mathfrak{T})_{[0]}$  is an extension as stated.  $\square$

**Proposition 4.3.3.** *The above extension is strict.*

*Proof:* We must show that

- i*) the inclusion  $\mathcal{I} : k[Z/Z'] \rightarrow U_q^\leq(\mathfrak{T})_{[0]}$  given by  $\mathcal{I} = f^{-1} \circ (1 \otimes -)$  is an algebra map and
- ii*) the projection  $\mathcal{P} : U_q^\leq(\mathfrak{T})_{[0]} \rightarrow U_q^\leq(\mathfrak{T})$  defined by  $\mathcal{P} = (\text{id} \otimes \varepsilon) \circ f$  is a coalgebra map.

For *i*), we have  $\mathcal{I}(Q_g) = f^{-1}(1 \otimes Q_g) = K_{\hat{j}(g)}$  where  $\hat{j} : Z/Z' \rightarrow Z$  is the splitting map. But  $\hat{j}$  is a group homomorphism, so  $\mathcal{I}$  is an algebra map. For *ii*), the relevant commutative diagram is

$$\begin{array}{ccc} U_q^\leq(\mathfrak{T})_{[0]} & \xrightarrow{\mathcal{P}} & U_q^\leq(\mathfrak{T}) \\ \Delta \downarrow & & \downarrow \Delta \\ U_q^\leq(\mathfrak{T})_{[0]} \otimes U_q^\leq(\mathfrak{T})_{[0]} & \xrightarrow{\mathcal{P} \otimes \mathcal{P}} & U_q^\leq(\mathfrak{T}) \otimes U_q^\leq(\mathfrak{T}) \end{array}$$

where  $\Delta$  is the coproduct on  $U_q^\leq(\mathfrak{T})_{[0]}$  and, by restriction, on  $U_q^\leq(\mathfrak{J})$ . We require the equality  $(\mathcal{P} \otimes \mathcal{P}) \circ \Delta = \Delta \circ \mathcal{P}$ .

Recall that  $\varepsilon : k[Z/Z'] \rightarrow k$  is given by  $\varepsilon(Q_g) = 1$  for all  $g \in Z/Z'$ , so, again writing  $f(y) = y' \otimes Q_{\deg y}$  for  $y \in U_q^\leq(\mathfrak{T})_{[0]}$ , we have  $\mathcal{P}(y) = y'$ . Now  $\mathcal{P}, \Delta$  are algebra maps so it is enough to show equality on generators. First, however, it is immediate that equality holds for elements of the sub-Hopf algebra  $U_q^\leq(\mathfrak{J})$ , since  $\mathcal{P} = \text{id}$  on  $U_q^\leq(\mathfrak{J})$ : for all  $y \in U_q^\leq(\mathfrak{J})$ ,  $f(y) = y \otimes Q_0$ . So, we need only check equality for  $K_\mu$ ,  $\mu \notin Z'$ . Then

$$\begin{aligned} ((\mathcal{P} \otimes \mathcal{P}) \circ \Delta)(K_\mu) &= (\mathcal{P} \otimes \mathcal{P})(K_\mu \otimes K_\mu) \\ &= K_{p_1(\mu)} \otimes K_{p_1(\mu)} \end{aligned}$$

and

$$\begin{aligned} (\Delta \circ \mathcal{P})(K_\mu) &= \Delta(K_{p_1(\mu)}) \\ &= K_{p_1(\mu)} \otimes K_{p_1(\mu)} \end{aligned}$$

as required. Here,  $p_1(\mu)$  is the element of  $Z'$  such that  $\mu = p_1(\mu) + p_2(\mu)$  (uniquely) with  $p_2(\mu) \in Z''$ ,  $Z = Z' \oplus Z''$ .  $\square$

**Theorem 4.3.4.**  $U_q^\leq(\mathfrak{T})_{[0]} \cong U_q^\leq(\mathfrak{J}) \otimes k[Z/Z']$  as Hopf algebras.

*Proof:* Firstly, by the above, we have  $U_q^\leq(\mathfrak{T})_{[0]} \cong U_q^\leq(\mathfrak{J}) \blacktriangleright\blacktriangleleft k[Z/Z']$  as Hopf algebras. This follows from [Maj02, Proposition 21.9]. The notation “ $\blacktriangleright\blacktriangleleft$ ” describes two simultaneous semi-direct products and coproducts,  $\bowtie$  and  $\blacktriangleleft$ . Now, the coaction of  $U_q^\leq(\mathfrak{J})$  on  $k[Z/Z']$  is the push-out of  $\Delta$  on  $U_q^\leq(\mathfrak{T})_{[0]}$  via  $\mathcal{P}$ , restricted to  $k[Z/Z']$ . That is,  $\gamma : k[Z/Z'] \rightarrow k[Z/Z'] \otimes U_q^\leq(\mathfrak{J})$ ,  $\gamma = (\text{id} \otimes \mathcal{P}) \circ \Delta \circ \mathcal{I}$ . We calculate  $\gamma$  to see that this is indeed a coaction on  $k[Z/Z']$ . Let  $Q_g \in k[Z/Z']$ . Then

$$\begin{aligned} ((\text{id} \otimes \mathcal{P}) \circ \Delta \circ \mathcal{I})(Q_g) &= ((\text{id} \otimes \mathcal{P}) \circ \Delta)(K_{\hat{j}(g)}) \\ &= (\text{id} \otimes \mathcal{P})(K_{\hat{j}(g)} \otimes K_{\hat{j}(g)}) \\ &= K_{\hat{j}(g)} \otimes K_{p_1(\hat{j}(g))} \\ &= K_{\hat{j}(g)} \otimes 1 \\ &= \mathcal{I}(Q_g) \otimes 1 \end{aligned}$$

since  $p_1(\hat{j}(g)) = 0$ :  $\hat{j}(g) \in Z''$ . So  $\gamma$  is the trivial coaction.

The action of  $k[Z/Z']$  on  $U_q^\leq(\mathfrak{J})$  is the pull-back of the adjoint action of  $U_q^\leq(\mathfrak{T})_{[0]}$  on itself via  $\mathcal{I}$ , restricted to  $U_q^\leq(\mathfrak{J})$ . That is,  $\alpha = \mathcal{P} \circ \text{Ad} \circ (\mathcal{I} \otimes \text{id})$ . We calculate  $\alpha$ . Let  $Q_g \otimes y \in k[Z/Z'] \otimes U_q^\leq(\mathfrak{J})$ . Then

$$\begin{aligned} (\mathcal{P} \circ \text{Ad} \circ (\mathcal{I} \otimes \text{id}))(Q_g \otimes y) &= (\mathcal{P} \circ \text{Ad})(K_{\hat{j}(g)} \otimes y) \\ &= \mathcal{P}(K_{\hat{j}(g)} y K_{\hat{j}(g)}^{-1}) \\ &= \mathcal{P}(K_{\hat{j}(g)}) \mathcal{P}(y) \mathcal{P}(K_{\hat{j}(g)})^{-1} \\ &= 1 \cdot \mathcal{P}(y) \cdot 1 \\ &= y \end{aligned}$$

since  $y \in U_q^\leq(\mathfrak{J})$  and  $\mathcal{P}|_{U_q^\leq(\mathfrak{J})} = \text{id}$ . In the above, we used the adjoint action as a map  $\text{Ad} : h \otimes g \mapsto h_{(1)} g S h_{(2)}$ , the group-like  $\Delta$  on the  $K_\mu$ ,  $S K_\mu = K_\mu^{-1}$  and the fact that  $\mathcal{P}$  is an algebra map. So,  $\alpha$  is also trivial.  $\square$

So, we see that  $U_q^\leq(\mathfrak{T})_{[0]}$  is a central extension of  $U_q^\leq(\mathfrak{J})$ , with the rank of the extension equal to that of  $Z/Z'$  as a free Abelian group, namely  $|I \setminus J|$ .

#### 4.4 $U_q(\mathfrak{T})$ is a double-bosonisation

Recall from Theorem 4.3.1 that we constructed  $B = B(\mathfrak{T}, \mathfrak{J}, \iota)$  in the category of  $D(U_q^\leq(\mathfrak{T})_{[0]})$ -modules. However, to reconstruct  $U_q(\mathfrak{T})$  as a double-bosonisation, we require  $B$  in the category of (left)  $\widetilde{U_q(\mathfrak{J})}$ -modules, where “ $\sim$ ” denotes a central extension. To see that this is indeed the case, we make use of the above analysis of the structure of  $U_q^\leq(\mathfrak{T})_{[0]}$  and define a projection  $D(U_q^\leq(\mathfrak{T})_{[0]}) \rightarrow \widetilde{U_q(\mathfrak{J})}$  whose kernel annihilates  $B$ .

**Theorem 4.4.1.** *Let  $U_q(\mathfrak{J}) \otimes k[Z/Z']$  have the tensor product Hopf algebra structure. Then  $U_q(\mathfrak{J}) \otimes k[Z/Z']$  is a quotient Hopf algebra of  $D(U_q^\leq(\mathfrak{T})_{[0]})$ .*

*Proof:* Recall that in Section 4.3, we described how to obtain  $\mathbb{Z}$ -gradings on  $U_q(\mathfrak{T})$  from maps  $\gamma : I \rightarrow \mathbb{Z}$ , by setting  $\deg E_i = -\deg F_i = \gamma(i)$  for  $i \in I$  and  $\deg K_\mu = 0$  for  $\mu \in Z$ . Consider such a map defined by  $i \mapsto 1$  for all  $i \in I$ . This therefore determines a  $\mathbb{Z}$ -grading

on  $U_q(\mathfrak{T})$ . Observe that we also have a  $\mathbb{Z}$ -grading on  $D(U_q^\leq(\mathfrak{T})) = U_q^\leq(\mathfrak{T}) \bowtie U_q^\geq(\mathfrak{T})$ , by similar formulæ:  $\deg(1 \otimes E_i) = -\deg(F_i \otimes 1) = 1$ ,  $\deg(K_\mu \otimes 1) = \deg(1 \otimes K_\mu) = 0$ .

Now, for  $i \neq 0$ ,  $D(U_q^\leq(\mathfrak{T}))_{[i]} \cong U_q(\mathfrak{T})_{[i]}$  (as vector spaces), via  $\mathbb{P}_i \stackrel{\text{def}}{=} \mathbb{P}|_{D(U_q^\leq(\mathfrak{T}))_{[i]}}$ , the restriction of Drinfel'd's projection, which is a graded map. However,  $\mathbb{P}_0$  has non-trivial kernel: this is precisely the ideal generated by elements of the form  $K_\mu \otimes K_\mu^{-1} - 1 \otimes 1$  for  $\mu \in Z$ .

We would like a projection  $D(U_q^\leq(\mathfrak{T})_{[0]}) \rightarrow U_q(\mathfrak{J}) \otimes k[Z/Z']$  and we make use of the above decomposition of  $U_q^\leq(\mathfrak{T})_{[0]}$  as  $U_q^\leq(\mathfrak{J}) \otimes k[Z/Z']$  and Drinfel'd's projection. Recall from the proof of Proposition 4.3.2 that the decomposition  $Z \cong Z' \oplus Z/Z'$  means that we may define maps  $p_1 : Z \rightarrow Z'$ ,  $p_2 : Z \rightarrow Z/Z'$  so that to  $\mu \in Z$  we have the unique associated pair  $(p_1(\mu), p_2(\mu))$ . Retaining our earlier notation, let  $\{Q_\alpha \mid \alpha \in Z/Z'\}$  be a generating set for  $k[Z/Z']$  ( $Q_0 = 1_{k[Z/Z']}$ ).

Since  $D(U_q^\leq(\mathfrak{T})_{[0]})$  is generated by  $\{F_j \otimes 1, 1 \otimes E_j, K_\mu \otimes 1, 1 \otimes K_\mu \mid j \in \iota(J), \mu \in Z\}$ , we may define  $\Phi : D(U_q^\leq(\mathfrak{T})_{[0]}) \rightarrow U_q(\mathfrak{J}) \otimes k[Z/Z']$  by

$$\begin{aligned} \Phi(F_j \otimes 1) &= F_j \otimes 1, \\ \Phi(1 \otimes E_j) &= E_j \otimes 1, \\ \Phi(K_\mu \otimes 1) &= K_{p_1(\mu)} \otimes Q_{p_2(\mu)}, \\ \Phi(1 \otimes K_\mu) &= K_{p_1(\mu)} \otimes Q_{p_2(\mu)}, \end{aligned}$$

extended linearly and multiplicatively. It is clear that this is a Hopf algebra projection.  $\square$

The kernel of this map  $\Phi$  is clearly generated by  $\{K_\mu \otimes K_\mu^{-1} - 1 \otimes 1 \mid \mu \in Z\}$ , since as for Drinfel'd's projection  $\mathbb{P}$  we identify  $K_\mu \otimes 1$  and  $1 \otimes K_\mu$  in the image. Hence the kernel of  $\Phi$  annihilates  $B$ , since the identified elements in the quotient  $K_\mu \otimes 1$  and  $1 \otimes K_\mu$  have equal (left) actions on  $B$ . To see this, first note that the left action of  $1 \otimes K_\mu \in 1 \otimes U_q^\leq(\mathfrak{T})_{[0]}^{\text{op}} \subseteq D(U_q^\leq(\mathfrak{T})_{[0]})$  is obtained from a right action of  $K_\mu^{-1} \in U_q^\leq(\mathfrak{T})_{[0]}$  (recall that  $S(K_\mu) = K_\mu^{-1}$ ). This right action is itself obtained by evaluation against (a push-out of) the left adjoint coaction of  $U_q^\leq(\mathfrak{T})_{[0]}$ , since  $U_q^\leq(\mathfrak{T})_{[0]}$  is self-dual. This last

fact and the invertibility of the antipode allow us to translate between the actions and coactions on either side. Hence  $B$  is a  $(U_q(\mathfrak{J}) \otimes k[Z/Z'])$ -module.

For brevity we will sometimes denote the central extension  $U_q(\mathfrak{J}) \otimes k[Z/Z']$  by  $\widetilde{U_q(\mathfrak{J})}$ . We conclude by showing that  $U_q(\mathfrak{T})$  is isomorphic to the double-bosonisation of  $B$  and its dual by  $\widetilde{U_q(\mathfrak{J})}$ .

**Theorem 4.4.2.** *Let  $\mathfrak{J} \subseteq_\iota \mathfrak{T}$  be a sub-root datum of  $\mathfrak{T}$ . Then*

$$U_q(\mathfrak{T}) \cong B \rtimes \widetilde{U_q(\mathfrak{J})} \ltimes (B')^{\text{op}}$$

as Hopf algebras, where  $B'$  is dually paired with  $B = B(\mathfrak{T}, \mathfrak{J}, \iota)$ .

*Proof:* We show that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker } \mathbb{P} & \xrightarrow{i_1} & D(U_q^\leq(\mathfrak{T})) & \xrightarrow{\mathbb{P}} & U_q(\mathfrak{T}) \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & \text{Ker } \pi_2 & \xrightarrow{i_2} & D(B \rtimes U_q^\leq(\mathfrak{T})_{[0]}, U_q^\geq(\mathfrak{T})_{[0]} \ltimes (B')^{\text{op}}) & \xrightarrow{\pi_2} & B \rtimes \widetilde{U_q(\mathfrak{J})} \ltimes (B')^{\text{op}} \longrightarrow 0 \end{array}$$

The rows of this diagram are exact sequences of Hopf algebras. The first expresses the fact that the quantized enveloping algebra  $U_q(\mathfrak{T})$  may be obtained as a quotient of the double, via Drinfel'd's projection  $\mathbb{P}$ .

The second combines the above results on the structure of  $U_q^\leq(\mathfrak{T})_{[0]}$  with the observation that one may obtain the double-bosonisation  $B \rtimes H \ltimes (B')^{\text{op}}$  as a quotient of a double built from two (single) bosonisations (cf. [Maj99, Theorem 6.2]).

We carried out the above analysis on  $U_q^\leq(\mathfrak{T})$ , to obtain a braided group  $B$  such that  $\beta_\leq : U_q^\leq(\mathfrak{T}) \xrightarrow{\sim} B \rtimes U_q^\leq(\mathfrak{T})_{[0]}$ . However, we could equally well start with the self-dual Hopf algebra  $U_q^\geq(\mathfrak{T})$  and obtain a braided group  $(B')^{\text{op}}$  in the braided category of right  $D(U_q^\geq(\mathfrak{T})_{[0]})$ -modules such that  $\beta_\geq : U_q^\geq(\mathfrak{T}) \xrightarrow{\sim} U_q^\geq(\mathfrak{T})_{[0]} \ltimes (B')^{\text{op}}$ . Furthermore,  $B$  and  $B'$  are dually paired braided groups, via Lusztig's pairing (as in Section 2.3.6).

The double we use is  $(B \rtimes U_q^\leq(\mathfrak{T})_{[0]}) \ltimes (U_q^\geq(\mathfrak{T})_{[0]} \ltimes (B')^{\text{op}})$  and the projection  $\pi_2$  is the map identifying generators  $1 \otimes K_\mu \otimes 1 \otimes 1$  and  $1 \otimes 1 \otimes K_\mu \otimes 1$ , similarly to Drinfel'd's projection above. We see from Theorem 4.4.1 that the quotient defined by  $\pi_2$  contains  $B \rtimes \widetilde{U_q(\mathfrak{J})} \otimes 1$  and  $1 \otimes \widetilde{U_q(\mathfrak{J})} \ltimes (B')^{\text{op}}$  as sub-Hopf algebras, as desired.



Observe also that the stated double-bosonisation is well-defined, as  $U_q(\mathfrak{T})$  has an associated weak quasitriangular system (see Section 2.3.6) and this restricts to  $\widetilde{U_q(\mathfrak{J})}$ . The cross-relation in the double-bosonisation is the quantized enveloping algebra defining relation **(R4)** (Definition 2.3.10), the commutation relation for  $E_i$  and  $F_j$ —this relation is also encoded in the cross-relations of the double (see for example [Jos95, Section 3.2]).

The map  $\gamma$  is defined to be the Hopf algebra map such that the second square commutes. Then all the maps in the diagram are Hopf algebra maps. We see that there exists an isomorphism  $\alpha : \text{Ker } \mathbb{P} \rightarrow \text{Ker } \pi_2$ . We also have an isomorphism  $\beta : D(U_q^\leq(\mathfrak{T})) \rightarrow (B \bowtie U_q^\leq(\mathfrak{T})_{[0]}) \bowtie (U_q^\geq(\mathfrak{T})_{[0]} \bowtie (B')^{\text{op}})$  induced by the above isomorphisms  $\beta_\leq$  and  $\beta_\geq$ . It follows that  $\gamma$  is an isomorphism of Hopf algebras.  $\square$

## 4.5 The structure of $B$

From our results on general braided groups  $B$  arising from split projections of graded Hopf algebras, we know that  $B = B(\mathfrak{T}, \mathfrak{J}, \iota)$  associated to  $\mathfrak{J} \subseteq_\iota \mathfrak{T}$  is both graded and an Ad-submodule of  $U_q^\leq(\mathfrak{T})$ . We now examine the module, algebra and braided-coalgebra structures of  $B$ .

For  $S$  a finite set, denote by  $S^\mathbb{N}$  the set of all finite sequences of elements of  $S$ , including the empty sequence,  $\emptyset$ . If  $\alpha \in S^\mathbb{N}$ ,  $l(\alpha)$  will denote the length of  $\alpha$ ;  $l(\emptyset) = 0$ . If  $i : S \hookrightarrow M$  is an injective map from  $S$  into a monoid  $M$ , we define the weight of  $\alpha \in S^\mathbb{N}$  to be  $\text{wt}(\alpha) = \sum_{j=1}^{l(\alpha)} i(\alpha_j)$ . We set  $\text{wt}(\emptyset) = 0$  (0 denoting the identity element of  $M$ ).

### 4.5.1 The algebra structure of $B$

From the general results, we know that  $B$  is a graded algebra; however the general results do not give us much more information about  $B$  than this. Since  $U_q(\mathfrak{J})$  is defined by generators and relations, we would like to have a presentation for  $B$ . As a first step, we may explicitly identify a set of generators of  $B$ , as follows.

**Theorem 4.5.1.** *Let  $A$  be the  $\widetilde{U_q(\mathfrak{J})}$ -submodule of  $B = B(\mathfrak{T}, \mathfrak{J}, \iota)$  generated by the set  $\{F_\gamma H_{\text{wt}(\gamma)} \mid \gamma \in D^\mathbb{N}\}$  and let  $\mathcal{A}$  be the subalgebra of  $B$  generated by  $A$ . Then  $\mathcal{A} = B$ .*

*Proof:* Recall from Section 4.2 that we have a Hopf algebra isomorphism

$$\begin{aligned}\Upsilon : U_q^\leq(\mathfrak{T}) &\rightarrow B \rtimes U_q^\leq(\mathfrak{T})_{[0]}, \\ \Upsilon(h) &= h_{(1)}((S \circ \iota \circ \pi)(h_{(2)})) \otimes h_{(3)}.\end{aligned}$$

We calculate  $\Upsilon$  on the generators of  $U_q^\leq(\mathfrak{T})$  and obtain

$$\begin{aligned}\Upsilon(F_i) &= \begin{cases} 1 \otimes F_i & \text{if } i \in \iota(J) \\ F_i H_i \otimes H_i^{-1} & \text{if } i \in D \end{cases} \\ \Upsilon(K_\mu) &= 1 \otimes K_\mu \quad \forall \mu \in Z.\end{aligned}$$

For  $\emptyset \in D^\mathbb{N}$ ,  $F_\emptyset H_{\text{wt}(\emptyset)} = H_0 = 1$  (by convention).

We wish to show that  $\Upsilon(U_q^\leq(\mathfrak{T})) \subseteq \mathcal{A} \rtimes U_q^\leq(\mathfrak{T})_{[0]}$ . Consider a monomial  $F_\alpha K_\mu$ ,  $\alpha \in I^\mathbb{N}$ ,  $\mu \in Z$ . Recall that monomials of this form are a basis for  $U_q^\leq(\mathfrak{T})$ . Then  $\Upsilon(F_\alpha K_\mu) = \Upsilon(F_\alpha)(1 \otimes K_\mu)$  and so we need only show that  $\Upsilon(F_\alpha) \in \mathcal{A} \rtimes U_q^\leq(\mathfrak{T})_{[0]}$ .

We proceed by induction on  $l(\alpha)$ . For  $l(\alpha) = 1$ , the above formulæ for  $\Upsilon(F_i)$  suffice. Assume now that  $\Upsilon(F_\alpha) \in \mathcal{A} \rtimes U_q^\leq(\mathfrak{T})_{[0]}$  for all  $\alpha \in I^\mathbb{N}$  with  $l(\alpha) = r$ , for some  $r$ . Let  $\beta \in I^\mathbb{N}$  with  $l(\beta) = r + 1$ . Then we may write  $F_\beta = F_\alpha F_i$  with  $\alpha \in I^\mathbb{N}$ ,  $l(\alpha) = r$  and  $i \in I$ . Write  $\Upsilon(F_\alpha) = x^{(1)} \otimes x^{(2)}$  in Sweedler notation, with  $x^{(1)} \in \mathcal{A}$  and  $x^{(2)} \in U_q^\leq(\mathfrak{T})_{[0]}$  by the induction hypothesis. We then have two cases:

i) if  $i \in \iota(J)$  then

$$\begin{aligned}\Upsilon(F_\beta) &= \Upsilon(F_\alpha)\Upsilon(F_i) \\ &= (x^{(1)} \otimes x^{(2)})(1 \otimes F_i) \\ &= x^{(1)} \otimes x^{(2)} F_i \\ &\in \mathcal{A} \otimes U_q^\leq(\mathfrak{T})_{[0]}.\end{aligned}$$

ii) if  $i \in D$  then

$$\begin{aligned}\Upsilon(F_\beta) &= \Upsilon(F_\alpha)\Upsilon(F_i) \\ &= (x^{(1)} \otimes x^{(2)})(F_i H_i \otimes H_i^{-1})\end{aligned}$$

$$\begin{aligned}
 &= x^{(1)} \text{Ad}_{\iota(x^{(2)})_{(1)}}(F_i H_i) \otimes x^{(2)}_{(2)} H_i^{-1} \\
 &\quad \text{(by the form of the product in } B \rtimes U_q^\leq(\mathfrak{T})_{[0]}) \\
 &\in \mathcal{A} \otimes U_q^\leq(\mathfrak{T})_{[0]}
 \end{aligned}$$

Note that  $\text{Ad}_{\iota(x^{(2)})_{(1)}}(F_i H_i) \in \mathcal{A}$  since  $x^{(2)} \in U_q^\leq(\mathfrak{T})_{[0]}$  and  $U_q^\leq(\mathfrak{T})_{[0]}$  is a sub-Hopf algebra of  $U_q^\leq(\mathfrak{T})$ .

Thus,  $\Upsilon(U_q^\leq(\mathfrak{T})) \subseteq \mathcal{A} \rtimes U_q^\leq(\mathfrak{T})_{[0]}$  but then  $\Upsilon(B) = B \otimes 1 \subseteq \mathcal{A} \otimes 1$  and hence  $B = \mathcal{A}$ .  $\square$

From this, the following is immediate.

**Corollary 4.5.2.** *The submodule  $B_1$ , which is the first graded component of  $B$ , is generated as a  $\widetilde{U_q(\mathfrak{J})}$ -module by the set  $\{F_d H_{\text{wt}(d)} \mid d \in D\}$ .*  $\square$

Observe that  $\mathcal{A}$  is graded, since  $\text{Ad}$  is a graded map, with  $A_n \subseteq B_n$  and  $A_1 = B_1$ .

**Corollary 4.5.3.**  *$B$  is generated as an algebra by  $\widetilde{B}_1 = B_1 \oplus B_0 = B_1 \oplus k1$ .*

*Proof:* This follows from the proof of the theorem—in particular, part *ii*) (the case  $i \in D$ ) and Lemma 4.2.8.  $\square$

## 4.5.2 The module structure of $B$

We would like some additional information on the module structure of  $B$ , in particular regarding its set of weights. Recall from Lemma 4.2.4 that  $B$  is an  $\text{Ad}$ -submodule of  $H = U_q^\leq(\mathfrak{T})$ . Although we want to know the module structure of  $B$  as a  $\widetilde{U_q(\mathfrak{J})}$ -module, we first consider the adjoint action of  $U_q^\leq(\mathfrak{T})$  on itself. For  $\alpha \in I^\mathbb{N}$ ,  $\mu \in Z$ , the weight of  $F_\alpha K_\mu$  for the adjoint action is  $\text{wt}(-\alpha) = -\sum_{j=1}^{l(\alpha)} i_2(\alpha_j)$ , where  $i_2 : I \hookrightarrow X$  is the injection of the index set  $I$  into the character lattice  $X$ . Since the  $F_\alpha K_\mu$  span  $U_q^\leq(\mathfrak{T})$ , the set of weights of  $U_q^\leq(\mathfrak{T})$  for  $\text{Ad}$  is  $-\mathbb{N}[i_2(I)]$ . That is,  $P_{\mathfrak{T}}(B) = -\mathbb{N}[i_2(I)]$ . Define  $\text{mult}_D(\alpha) = |\{\alpha_j \mid j \in D\}|$  for  $\alpha \in I^\mathbb{N}$  and  $\text{mult}_D(\text{wt}(-\alpha)) = \text{mult}_D(\alpha)$ . Note that  $\text{mult}_D(\alpha) = \deg F_\alpha$  for the grading described at the start of Section 4.3.

We have that  $F_\gamma H_{\text{wt}(\gamma)} \in B$  for  $\gamma \in D$ ; for brevity, write  $F_d H_d$  for  $F_d H_{\text{wt}(d)}$ . By the above,  $F_d H_d$  has weight  $\text{wt}(-i_2(d))$ . Now  $\text{wt}(-i_2(d))$  is not in general  $I$ -dominant

(see Definition 2.3.12):  $\langle i_1(d), -i_2(d) \rangle = -C_{dd} = -2$ . However, its image under  $\rho$  is  $J$ -dominant.

**Lemma 4.5.4.** *Let  $d \in D$ . Set  $\lambda'_d = \rho(-i_2(d)) \in X'$ , where  $\rho$  is the restriction map of Definition 4.1.3. Then  $\lambda'_d$  is  $J$ -dominant.*

*Proof:* For  $j \in J$  we have

$$\begin{aligned} \langle i'_1(j), \lambda'_d \rangle' &= \langle i'_1(j), \rho(-i_2(d)) \rangle' \\ &= \langle i'_1(j), -i_2(d) \rangle \\ &= -\langle i_1(\iota(j)), i_2(d) \rangle \\ &= -C_{\iota(j)d} \\ &\geq 0 \end{aligned}$$

since  $\iota(j) \neq d$  ( $I = \iota(J) \sqcup D$ ). □

Since the map  $i_2 : I \hookrightarrow X$  is injective, we have the decomposition

$$i_2(I) = i_2(\iota(J)) \sqcup i_2(D) = i'_2(J) \sqcup i_2(D)$$

(where  $\sqcup$  denotes the disjoint union). Let  $\lambda \in P_{\mathfrak{T}}(B)$  and  $S \subseteq I$ . Let  $\lambda_S$  denote  $-\sum_{k \in S} n_k i_2(k)$  where  $\lambda = -\sum_{k \in I} n_k i_2(k) = \lambda_I$ ,  $n_k \in \mathbb{N}$ . Then  $\lambda = \lambda_{\iota(J)} + \lambda_D$ , uniquely.

Next we consider primitive vectors for the action of  $U_q(\mathfrak{J})$ , which we recall to be vectors  $b \in B$  such that  $E_j \triangleright b = 0$  for all  $j \in J$ .

**Lemma 4.5.5.** *The element  $F_\gamma H_{\text{wt}(\gamma)}$ ,  $\gamma \in D^{\mathbb{N}}$ , is a primitive vector for the action of  $\widetilde{U_q(\mathfrak{J})}$  on  $B$ , of weight  $\rho(-\text{wt}(\gamma))$ .*

*Proof:* For all  $j \in J$  we have

$$\begin{aligned} E_{\iota(j)} \triangleright F_\gamma H_{\text{wt}(\gamma)} &= \text{Ad}_{E_{\iota(j)}} F_\gamma H_{\text{wt}(\gamma)} \\ &= E_{\iota(j)} F_\gamma H_{\text{wt}(\gamma)} - H_{\iota(j)} F_\gamma H_{\text{wt}(\gamma)} H_{\iota(j)}^{-1} E_{\iota(j)} \\ &= E_{\iota(j)} F_\gamma H_{\text{wt}(\gamma)} - q^{-\iota(j) \cdot \text{wt}(\gamma)} F_\gamma H_{\text{wt}(\gamma)} E_{\iota(j)} \end{aligned}$$

$$\begin{aligned}
 &= E_{\iota(j)} F_{\gamma} H_{\text{wt}(\gamma)} - q^{-\iota(j) \cdot \text{wt}(\gamma)} q^{-((- \text{wt}(\gamma)) \cdot \iota(j))} F_{\gamma} E_{\iota(j)} H_{\text{wt}(\gamma)} \\
 &= [E_{\iota(j)}, F_{\gamma}] H_{\text{wt}(\gamma)} \quad \text{since } \cdot \text{ is symmetric} \\
 &= 0 \quad \text{by (R4): } \gamma_i \neq \iota(j) \text{ for all } i.
 \end{aligned}
 \quad \square$$

We have the preorder  $\prec$  on  $X'$  given by  $\mu \prec \nu$  if and only if  $\nu - \mu \in \mathbb{N}[i'_2(J)]$ . Hence, we have the following.

**Lemma 4.5.6.** *For all  $\lambda \in P_{\mathfrak{T}}(B)$ ,  $\rho(\lambda) \prec \rho(\lambda_D)$  in  $X'$ .*

*Proof:* We have  $\lambda - \lambda_D = \lambda_{\iota(J)} \in -\mathbb{N}[i'_2(J)] \subseteq X'$  (by definition,  $i'_2 = i_2 \circ \iota$ ) and so  $\rho(\lambda - \lambda_D) = \lambda - \lambda_D$  ( $\rho|_{X'} = \text{id}_{X'}$ ). Then  $\lambda_{\iota(J)} \prec 0$  implies  $\rho(\lambda) - \rho(\lambda_D) = \rho(\lambda - \lambda_D) \prec 0$ , i.e.  $\rho(\lambda) \prec \rho(\lambda_D)$ .  $\square$

Now, if  $\lambda = -\sum_{k \in I} n_k i_2(k)$ ,  $n_k \in \mathbb{N}$ , then

$$\rho(\lambda_D) = \sum_{k \in D} n_k \rho(-i_2(k)) = \sum_{k \in D} n_k \lambda'_k$$

in the notation of Lemma 4.5.4. Set  $\lambda'_D = \rho(\lambda_D)$ . It then follows from Lemma 4.5.4 that  $\lambda'_D$  is  $J$ -dominant.

We may also consider the action of the Weyl group,  $W_{\mathfrak{J}}$ , associated to  $\mathfrak{J}$  on  $X'$  and its relationship with  $\prec$  and  $\rho$ . Recall that  $W_{\mathfrak{J}}$  is generated by the simple reflections  $\{\sigma_{\iota(j)} \mid j \in J\}$ . In particular,  $W_{\mathfrak{J}}$  is a subgroup of  $W_{\mathfrak{T}}$ , the Weyl group associated to  $\mathfrak{T}$ .

**Lemma 4.5.7.** *Let  $\lambda \in P_{\mathfrak{T}}(B)$ . For all  $\sigma \in W_{\mathfrak{J}}$ ,  $\sigma(\rho(\lambda)) \prec \lambda'_D = \rho(\lambda_D)$ . That is, all Weyl conjugates of  $\rho(\lambda)$  are also less than  $\lambda'_D$  with respect to  $\prec$ .*

*Proof:* We may restrict to considering the simple reflections  $\sigma_{\iota(j)}$ ,  $j \in J$ . For  $j \in J$ , the action of  $\sigma_{\iota(j)}$  on  $\mu \in X'$  is

$$\begin{aligned}
 \sigma_{\iota(j)}(\mu) &= \mu - \langle i_1(\iota(j)), \mu \rangle' i_2(\iota(j)) \\
 &= \mu - \langle i'_1(j), \mu \rangle' i'_2(j).
 \end{aligned}$$

Similarly, for  $k \in I$ , the action of  $\sigma_k$  on  $\nu \in X$  is

$$\sigma_k(\nu) = \nu - \langle i_1(k), \nu \rangle i_2(k).$$

The result then follows by showing that for  $\lambda \in X$ ,  $\rho(\sigma_{\iota(j)}(\lambda)) = \sigma_{\iota(j)}(\rho(\lambda))$ , for we may then use Lemma 4.5.6.

$$\begin{aligned}
 \rho(\sigma_{\iota(j)}(\lambda)) &= \rho(\lambda - \langle i_1(\iota(j)), \lambda \rangle i_2(\iota(j))) \\
 &= \rho(\lambda) + \langle i'_1(j), \lambda \rangle \rho(-i'_2(j)) \\
 &= \rho(\lambda) + \langle i'_1(j), \rho(\lambda) \rangle' \rho(-i'_2(j)) \\
 &= \rho(\lambda) + \langle i'_1(j), \rho(\lambda) \rangle' (-i'_2(j)) \\
 &= \rho(\lambda) - \langle i'_1(j), \rho(\lambda) \rangle' i'_2(j) \\
 &= \sigma_{\iota(j)}(\rho(\lambda)).
 \end{aligned}$$

We have  $\rho(-i'_2(j)) = -i'_2(j)$  since  $-i'_2(j) \in X'$  and  $\rho|_{X'} = \text{id}_{X'}$ . From the definition of the action of  $\sigma_{\iota(j)}$  we see that  $(\sigma_{\iota(j)}(\lambda))_D = \lambda_D$ .

Explicitly, consider  $\sigma_{\iota(j)}(\lambda) - \lambda_D$ . We have

$$\sigma_{\iota(j)}(\lambda) - \lambda_D = \lambda - \langle i_1(\iota(j)), \lambda \rangle i_2(\iota(j)) - \lambda_D \in -\mathbb{N}[i'_2(J)]$$

since  $\lambda - \lambda_D = \lambda_{\iota(j)} \in -\mathbb{N}[i'_2(J)]$  and  $i_2(\iota(j)) = i'_2(j) \in i'_2(J)$ . Consequently, we have  $\rho(\sigma_{\iota(j)}(\lambda) - \lambda_D) = \sigma_{\iota(j)}(\lambda) - \lambda_D \in -\mathbb{N}[i'_2(J)]$ . From the above and the homomorphism property of  $\rho$ ,

$$\begin{aligned}
 \rho(\sigma_{\iota(j)}(\lambda) - \lambda_D) &= \rho(\sigma_{\iota(j)}(\lambda)) - \rho(\lambda_D) \\
 &= \sigma_{\iota(j)}(\rho(\lambda)) - \rho(\lambda_D).
 \end{aligned}$$

Combining these,  $\sigma_{\iota(j)}(\rho(\lambda)) - \rho(\lambda_D) \prec 0$  and hence  $\sigma_{\iota(j)}(\rho(\lambda)) \prec \rho(\lambda_D)$ .  $\square$

Thus far, we have not used the grading on  $B$ . Combining Lemma 4.2.7 and Section 4.4, we have that  $B_n$  is a  $U_q(\mathfrak{J})$ -submodule of  $B$  for all  $n \in \mathbb{N}$ . So, we first turn our attention to  $B_1$ . Since  $B_0 = k$  (Lemma 4.2.8),  $B_0$  is the trivial  $U_q(\mathfrak{J})$ -module.

By Corollary 4.5.2,  $B_1$  is generated as a  $\widetilde{U_q(\mathfrak{J})}$ -module by its primitive vectors, namely  $\{F_d H_d \mid d \in D\}$ . Let  $V(\lambda'_d)$  be the submodule of  $B_1$  generated by  $F_d H_d$ . We remark that although  $V(\lambda'_{d_1}) \cap V(\lambda'_{d_2}) = 0$  for  $d_1 \neq d_2$ , we may have  $V(\lambda'_{d_1}) \cong V(\lambda'_{d_2})$  as  $\widetilde{U_q(\mathfrak{J})}$ -modules. Indeed, possibly  $\lambda'_{d_1} = \lambda'_{d_2}$ .

For a weight  $\lambda' \in X'$ , recall from Section 2.3.6 that we have  $M(\lambda')$ , the Verma module with highest weight vector  $v_{\lambda'}$  of weight  $\lambda'$ . Recall also its universal property, that for any  $M \in {}_{\mathfrak{g}}\mathcal{M}^{\text{wt}}$  and any primitive vector  $m_{\lambda'} \in M$  of weight  $\lambda'$ , there exists a unique morphism (of weight modules)  $t : M(\lambda') \rightarrow M$  such that  $t(v_{\lambda'}) = m_{\lambda'}$ . Let  $d \in D$ . By definition,  $V(\lambda'_d)$  above is generated by a primitive vector  $F_d H_d$ , so we have the unique morphism  $t_d : M(\lambda'_d) \rightarrow V(\lambda'_d)$  and furthermore,  $t_d$  is onto. (We have  $t_d(v_{\lambda'_d}) = F_d H_d$  but we do not need this.)

Now  $t_d$  factors through the associated Weyl module,  $L(\lambda'_d)$ , which is integrable. As in the proof of [Lus93, Proposition 3.5.8], we need only see that  $\lambda'_d$  is dominant and that  $F_{\iota(j)}^{<i'_1(j), \lambda'_d>'+1} \triangleright F_d H_d = 0$  for all  $j \in J$ . The first is given by Lemma 4.5.4 and the second follows immediately from the  $q$ -Serre relations. (See [Jan96, Lemma 4.18] and the following sections for more details.) Let  $\hat{t}_d : L(\lambda'_d) \rightarrow V(\lambda'_d)$  denote the induced morphism. Since  $t_d$  is onto,  $\hat{t}_d$  is also onto. Hence:

**Proposition 4.5.8.**  *$V(\lambda'_d)$  is integrable, therefore  $B_1$  is integrable. If  $L(\lambda'_d)$  is finite-dimensional for all  $d \in D$  then  $V(\lambda'_d)$  and  $B_1$  are finite-dimensional. If  $L(\lambda'_d)$  is simple then  $V(\lambda'_d) \cong L(\lambda'_d)$ .*  $\square$

Recall that for  $\lambda \in -\mathbb{N}[i_2(I)]$ , we have a (unique) decomposition  $\lambda = \lambda_{\iota(J)} + \lambda_D$  where  $\lambda_{\iota(J)} \in -\mathbb{N}[i_2(\iota(J))]$  and  $\lambda_D \in -\mathbb{N}[i_2(D)]$ . Then we have  $\text{wt}(\gamma) = (\text{wt}(\gamma))_D$  for all  $\gamma \in D^{\text{fl}}$  —  $\text{wt}(\gamma)$  has no part in  $-\mathbb{N}[i'_2(J)]$ . Hence  $\rho((- \text{wt}(\gamma))_D) = \rho(- \text{wt}(\gamma))$ ; set  $\rho((- \text{wt}(\gamma))_D) = \lambda'_\gamma$ . So, by Lemma 4.5.6 we have for any  $\lambda \in X$  with  $\text{mult}_D(\lambda) = n$ ,  $\rho(\lambda) \prec \lambda'_\gamma$  for some  $\gamma \in D^{\mathbb{N}}$ .

By Theorem 4.5.1 the submodules  $B_n$ ,  $n \geq 2$ , are direct sums of submodules of tensor products of the  $V(\lambda'_d)$ . Hence we may deduce the following.

**Theorem 4.5.9.**  *$B$  is integrable, as a direct sum of the  $B_n$ , which are integrable, and  $B$  is a direct sum of quotients of Weyl modules and tensor products of these.*  $\square$

We observe that these remarks also apply to  $\bigoplus_{n \leq N} B_n$  for any  $N \in \mathbb{N}$ . That is,  $\bigoplus_{n \leq N} B_n$  is an integrable  $\widetilde{U_q(\mathfrak{J})}$ -module, though not necessarily finite-dimensional.

### 4.5.3 The coalgebra structure of $B$

We turn to the coalgebra structure of the braided group  $B = B(\mathfrak{T}, \mathfrak{J}, \iota)$ .

**Definition 4.5.1.** Let  $(B, \underline{m}, \underline{\eta}, \underline{\Delta}, \underline{\varepsilon}, \underline{S})$  be a Hopf algebra in a braided category. We say  $b \in B$  is *braided-primitive* if  $\underline{\Delta}b = b \underline{\otimes} 1 + 1 \underline{\otimes} b$ . We will denote the vector space of braided-primitive elements of  $B$  by  $\underline{\text{Prim}}(B)$ .

We note that  $\underline{\text{Prim}}(B)$  is *not* a subalgebra of  $B$ . The next two lemmas give us information on the space of braided-primitive elements in the braided group associated to a sub-root datum.

**Lemma 4.5.10.** Let  $B = B(\mathfrak{T}, \mathfrak{J}, \iota)$  and let  $b \in \underline{\text{Prim}}(B)$ . Then for any  $x \in U_q(\mathfrak{J})$ , we have  $\text{Ad}_x(b) \in \underline{\text{Prim}}(B)$ .

*Proof:* By definition,  $\underline{\Delta}$  is Ad-covariant, as it is a morphism in the module category. Then

$$\begin{aligned}
 \underline{\Delta}(\text{Ad}_x(b)) &= \text{Ad}_{x_{(1)}}(b_{(1)}) \otimes \text{Ad}_{x_{(2)}}(b_{(2)}) \\
 &= \text{Ad}_{x_{(1)}}(b) \otimes \text{Ad}_{x_{(2)}}(1) + \text{Ad}_{x_{(1)}}(1) \otimes \text{Ad}_{x_{(2)}}(b) \\
 &= \text{Ad}_{x_{(1)}}(b) \otimes \varepsilon(x_{(2)}) + \varepsilon(x_{(1)}) \otimes \text{Ad}_{x_{(2)}}(b) \\
 &= x_{(1)}bSx_{(2)} \otimes \varepsilon(x_{(3)}) + \varepsilon(x_{(1)}) \otimes x_{(2)}bSx_{(3)} \\
 &= x_{(1)}bSx_{(2)}\varepsilon(x_{(3)}) \otimes 1 + 1 \otimes \varepsilon(x_{(1)})x_{(2)}bSx_{(3)} \\
 &= x_{(1)}bSx_{(2)} \otimes 1 + 1 \otimes x_{(1)}bSx_{(2)} \\
 &= \text{Ad}_x(b) \otimes 1 + 1 \otimes \text{Ad}_x(b) \\
 &\in \underline{\text{Prim}}(B)
 \end{aligned}$$

□

**Lemma 4.5.11.** For  $d \in D$ ,  $F_dH_d \in \underline{\text{Prim}}(B)$ .

*Proof:* Recall that  $\underline{\Delta}b = b_{(1)}(S \circ \iota \circ \pi)(b_{(2)}) \otimes b_{(3)}$ . A simple calculation shows that

$$\Delta^2(F_dH_d) = F_dH_d \otimes 1 \otimes 1 + H_d \otimes F_dH_d \otimes 1 + H_d \otimes H_d \otimes F_dH_d.$$

Hence,

$$\begin{aligned}
 \underline{\Delta}(F_dH_d) &= F_dH_d \cdot S(1) \otimes 1 + H_d \cdot 0 \otimes 1 + H_d \cdot H_d^{-1} \otimes F_dH_d \\
 &= F_dH_d \otimes 1 + 1 \otimes F_dH_d
 \end{aligned}$$



so indeed  $F_d H_d \in \underline{\text{Prim}}(B)$ .  $\square$

We deduce that  $B_1 \subseteq \underline{\text{Prim}}(B)$ , since by Corollary 4.5.2,  $B_1$  is spanned by elements given by  $\text{Ad}$  acting on the  $F_d H_d$ ,  $d \in D$ . However,  $B_1 \neq \underline{\text{Prim}}(B)$  in general and it remains to determine  $\underline{\text{Prim}}(B)$ . We comment on this further in Section 6.2.

## Chapter 5

# The triple construction

We study the triple of a quasitriangular Lie bialgebra as a natural extension of the Drinfel'd double. The triple is itself a quasitriangular Lie bialgebra. We prove several results about the algebraic structure of the triple, analogous to known results for the double. Among them, we prove that in the factorisable case the triple is isomorphic to a twisting of  $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$  by a certain cocycle. We also consider real forms of the triple and the triangular case.

### 5.1 The triple of a Lie bialgebra

We wish to discuss a special case of the double-bosonisation theorem, as recalled in Section 2.2.4.

Firstly, we have the third description of the Drinfel'd double (Section 2.2.3), now as a single bosonisation, as follows. We let  $\mathfrak{g}$  be a finite-dimensional quasitriangular Lie bialgebra and  $\underline{\mathfrak{g}}^*$  the dual of its transmutation. The bosonisation  $\underline{\mathfrak{g}}^* \bowtie \mathfrak{g}$  is isomorphic as a Lie bialgebra to the Drinfel'd double  $D(\mathfrak{g})$  ([Maj00, Example 3.9]).

In the factorisable case, the dual  $\underline{\mathfrak{g}}^*$  here can of course be replaced by  $\underline{\mathfrak{g}}$ . As a Lie algebra, we have a semidirect sum by definition of the bosonisation and furthermore we can easily see that a semidirect sum can be re-diagonalised to a direct sum. The coalgebra structure on this direct sum induced by these isomorphism is in fact precisely the one giving the double as a twisting (Theorem 2.2.1). Note that we also have the alternative description  $D(\mathfrak{g}) \cong \mathfrak{g} \ltimes \underline{\mathfrak{g}}^{*\text{op}}$ . This isomorphism will be described and used in Section 5.3.

We now define the triple, as a double-bosonisation using the transmutation of  $\mathfrak{g}$ . Let  $\mathfrak{g}$  be a finite-dimensional Lie bialgebra over a field  $k$  of characteristic not 2.

**Definition 5.1.1.** *In Theorem 2.2.2, set  $\mathfrak{b} = \underline{\mathfrak{g}}$  the transmutation of  $\mathfrak{g}$  and  $\mathfrak{c} = \underline{\mathfrak{g}}^*$ . We have  $\underline{\mathfrak{g}} \in {}_{\mathfrak{g}}\mathcal{M}$  by  $\text{ad}$ . Define  $T(\mathfrak{g}) = \underline{\mathfrak{g}} \rtimes \mathfrak{g} \ltimes \underline{\mathfrak{g}}^{\text{op}}$ , as a Lie bialgebra.*

The “ $T$ ” stands for “triple”: we will show later the comparison with the Drinfel’d double  $D(\mathfrak{g})$ . We have the Lie bialgebra structure of  $T(\mathfrak{g})$  given explicitly in terms of the brackets and cobrackets on  $\mathfrak{g}$ ,  $\underline{\mathfrak{g}}$  and the module structures for  $\underline{\mathfrak{g}}$ . We will show that these formulæ simplify.

We now restrict to the case of  $\mathfrak{g}$  factorisable, so that we can replace  $\mathfrak{c} = \underline{\mathfrak{g}}^*$  in the above definition by  $\mathfrak{c} = \underline{\mathfrak{g}}$ . The pairing we use,  $\ll, \gg$ , is the Killing form  $K : \mathfrak{g} \otimes \mathfrak{g} \rightarrow k$  which pairs  $\mathfrak{b} = \underline{\mathfrak{g}}$  and  $\mathfrak{c} = \underline{\mathfrak{g}}$  as braided-Lie bialgebras.

For clarity, we will refer to the three pieces from left to right in the definition of  $T(\mathfrak{g}) = \underline{\mathfrak{g}} \rtimes \mathfrak{g} \ltimes \underline{\mathfrak{g}}^{\text{op}}$  as  $\mathfrak{b}$ ,  $\mathfrak{g}$  and  $\mathfrak{c}^{\text{op}}$ , respectively. We will indicate the bracket and cobracket in  $T(\mathfrak{g})$  by a subscript “ $T$ ”, to distinguish it from the brackets and cobrackets of the individual pieces, which will carry a subscript  $\mathfrak{b}, \mathfrak{g}, \mathfrak{c}$  or  $\mathfrak{c}^{\text{op}}$  as appropriate.

**Lemma 5.1.1.** *As Lie algebras, let  $\mathfrak{b} = \underline{\mathfrak{g}} = \mathfrak{c}$ . The Lie bialgebra structure of the triple  $T(\mathfrak{g}) = \mathfrak{b} \rtimes \mathfrak{g} \ltimes \mathfrak{c}^{\text{op}}$  is given by:*

$$\begin{aligned} [b_1, b_2]_T &= [b_1, b_2]_{\mathfrak{b}} & [g, b]_T &= \alpha([g, \alpha^{-1}(b)]_{\mathfrak{g}}) \\ [g_1, g_2]_T &= [g_1, g_2]_{\mathfrak{g}} & [g, c]_T &= \varepsilon([g, \varepsilon^{-1}(c)]_{\mathfrak{g}}) \\ [c_1, c_2]_T &= -[c_1, c_2]_{\mathfrak{c}} \\ [b, c]_T &= -[b, \alpha \circ \beta^{-1}(c)]_{\mathfrak{b}} + [\alpha^{-1}(b), \beta^{-1}(c)]_{\mathfrak{g}} + [\beta \circ \alpha^{-1}(b), c]_{\mathfrak{c}} \\ \delta_T b &= \delta b + (r^{(1)} \triangleright b) \otimes (\alpha(r^{(2)}) - r^{(2)}) - (\alpha(r^{(2)}) - r^{(2)}) \otimes (r^{(1)} \triangleright b) \\ \delta_T g &= \delta g \\ \delta_T c &= \delta c + (\beta(r^{(1)}) - r^{(1)}) \otimes (r^{(2)} \triangleright c) - (r^{(2)} \triangleright c) \otimes (\beta(r^{(1)}) - r^{(1)}) \end{aligned}$$

for  $b, b_i \in \mathfrak{b}$ ,  $g, g_i \in \mathfrak{g}$  and  $c, c_i \in \mathfrak{c}$ . Here,  $\alpha, \beta, \varepsilon$  are the identity map between the pieces as detailed below.

*Proof:* We will use several isomorphisms between the three pieces in what follows. We set

$$\begin{aligned}\alpha : \mathfrak{g} &\rightarrow \mathfrak{b} = \underline{\mathfrak{g}}, & \alpha &= \text{id} \\ \beta : \mathfrak{g} &\rightarrow \mathfrak{c} = \underline{\mathfrak{g}}, & \beta &= \text{id} \\ \gamma : \mathfrak{c} &\rightarrow \mathfrak{c}^{\text{op}}, & \gamma &= \text{id} \\ \bar{\gamma} : \mathfrak{c} &\rightarrow \mathfrak{c}^{\text{op}}, & \bar{\gamma} : c &\mapsto -c \\ \varepsilon : \mathfrak{g} &\rightarrow \mathfrak{c}^{\text{op}}, & \varepsilon &= \gamma \circ \beta = \text{id} \quad \text{and} \\ \bar{\varepsilon} : \mathfrak{g} &\rightarrow \mathfrak{c}^{\text{op}}, & \bar{\varepsilon} &= \bar{\gamma} \circ \beta = -\varepsilon = -\text{id}.\end{aligned}$$

All of these except  $\gamma$  and  $\varepsilon$  are Lie algebra isomorphisms. We can now write the pairing of  $\mathfrak{b}$  and  $\mathfrak{c}$  explicitly as

$$\ll , \gg : \mathfrak{b} \otimes \mathfrak{c} \rightarrow k, \quad \ll b, c \gg = K(\alpha^{-1}(b), \beta^{-1}(c))$$

for  $b \in \mathfrak{b}$ ,  $c \in \mathfrak{c}$ .

Firstly,  $\mathfrak{g}$  has (unbraided) Lie bialgebra bracket and cobracket structures by assumption: we will denote these by plain brackets,  $[ , ]$ , and  $\delta$  respectively. The braided-Lie bialgebra structure for  $\underline{\mathfrak{g}}$  is that described in Definition 2.2.3. The braided-cobracket is

$$\underline{\delta}b = \alpha(2r_+^{(1)}) \otimes [b, \alpha(r_+^{(2)})]_{\mathfrak{b}},$$

for  $b \in \mathfrak{b} = \underline{\mathfrak{g}}$ , where  $r$  is the quasitriangular structure on  $\mathfrak{g}$  and  $2r_+$  is its symmetric part. Finally, the braided-Lie bialgebra structure of  $\mathfrak{c} = \underline{\mathfrak{g}}$  is the same as that of  $\mathfrak{b}$ , so

$$\begin{aligned}[ , ]_{\mathfrak{c}} &= [ , ]_{\mathfrak{b}} \\ &= [ , ]_{\mathfrak{g}} \quad \text{and} \\ \underline{\delta}c &= \beta(2r_+^{(1)}) \otimes [c, \beta(r_+^{(2)})]_{\mathfrak{c}}\end{aligned}$$

Each piece appears as a Lie subalgebra, so we need now to clarify the brackets between the pieces. We have  $\underline{\mathfrak{g}} \in {}_{\mathfrak{g}}\mathcal{M}$  by  $g \triangleright b = \text{ad}_g(b)$  for  $g \in \mathfrak{g}$ ,  $b \in \underline{\mathfrak{g}}$ , that is, the adjoint action. To be even more explicit, for  $g \in \mathfrak{g}$ ,  $b \in \mathfrak{b}$  and  $c \in \mathfrak{c}^{\text{op}}$  we have

$$\begin{aligned}g \triangleright b &= [\alpha(g), b]_{\mathfrak{b}} \\ &= \alpha([g, \alpha^{-1}(b)]_{\mathfrak{g}})\end{aligned}$$

and

$$\begin{aligned}
 g \triangleright c &= [\bar{\varepsilon}(g), c]_{\mathfrak{c}^{\text{op}}} \\
 &= \bar{\varepsilon}([g, \bar{\varepsilon}^{-1}(c)]_{\mathfrak{g}}) \\
 &= -\varepsilon([g, -\varepsilon^{-1}(c)]_{\mathfrak{g}}) \\
 &= \varepsilon([g, \varepsilon^{-1}(c)]_{\mathfrak{g}}).
 \end{aligned}$$

We set  $[g, b]_T = g \triangleright b$  and  $[g, c]_T = g \triangleright c$ , so the action of  $\mathfrak{g}$  on  $\mathfrak{b}$  and  $\mathfrak{c}^{\text{op}}$  is the adjoint action, with the bracket taken in  $\mathfrak{g}$  after the appropriate isomorphism has been applied.

These brackets come from the Lie algebra structure of the (single) bosonisations  $\mathfrak{b} \rtimes \mathfrak{g}$  and  $\mathfrak{g} \ltimes \mathfrak{c}^{\text{op}}$  as in [Maj00, Theorem 3.5]. Note that there are two minus signs which cancel, one from the reversed action in  $\mathfrak{g} \ltimes \mathfrak{c}^{\text{op}}$  and one from the “op”, so we just see the adjoint action of  $\mathfrak{g}$  on  $\mathfrak{c}$ .

For the double-bosonisation, the remaining bracket is the one between  $\mathfrak{b}$  and  $\mathfrak{c}$ . This is given by

$$[b, c] = \underline{b}_{(1)} \langle c, \underline{b}_{(2)} \rangle + \underline{c}_{(1)} \langle \underline{c}_{(2)}, b \rangle + 2r_+^{(1)} \langle c, r_+^{(2)} \rangle \triangleright b$$

for  $b \in \mathfrak{b}$ ,  $c \in \mathfrak{c}$ . Using the above definitions of  $\underline{\delta}$  and the pairing  $\ll, \gg$ , we obtain the following:

$$\begin{aligned}
 [b, c]_T &= \underline{b}_{(1)} \ll c, \underline{b}_{(2)} \gg + \underline{c}_{(1)} \ll \underline{c}_{(2)}, b \gg + 2r_+^{(1)} \ll c, r_+^{(2)} \triangleright b \gg \\
 &= \alpha(2r_+^{(1)}) \ll [b, \alpha(r_+^{(2)})]_{\mathfrak{b}}, c \gg + \beta(2r_+^{(1)}) \ll b, [c, \beta(r_+^{(2)})]_{\mathfrak{c}} \gg \\
 &\quad + 2r_+^{(1)} \ll \alpha([r_+^{(2)}, \alpha^{-1}(b)]_{\mathfrak{g}}), c \gg
 \end{aligned}$$

so by the ad-invariance of  $2r_+$ ,

$$\begin{aligned}
 [b, c]_T &= [\alpha(2r_+^{(1)}), b]_{\mathfrak{b}} \ll \alpha(r_+^{(2)}), c \gg + [\beta(2r_+^{(1)}), c]_{\mathfrak{c}} \ll b, \beta(r_+^{(2)}) \gg \\
 &\quad - [2r_+^{(1)}, \alpha^{-1}(b)]_{\mathfrak{g}} \ll \alpha(r_+^{(2)}), c \gg \\
 &= [\alpha(2r_+^{(1)})K(\alpha^{-1}(\alpha(r_+^{(2)})), \beta^{-1}(c)), b]_{\mathfrak{b}} \\
 &\quad + [\beta(2r_+^{(1)})K(\alpha^{-1}(b), \beta^{-1}(\beta(r_+^{(2)}))), c]_{\mathfrak{c}} \\
 &\quad - [2r_+^{(1)}K(\alpha^{-1}(\alpha(r_+^{(2)})), \beta^{-1}(c)), \alpha^{-1}(b)]_{\mathfrak{g}}.
 \end{aligned}$$

Then, using the quasitriangular expression for the inverse of the Killing form, that is,  $K^{-1}(\varphi) = 2r_+^{(1)}\varphi(r_+^{(2)})$ ,

$$\begin{aligned}
[b, c]_T &= [K^{-1}(K(\beta^{-1}(c))), b]_{\mathfrak{b}} + [K^{-1}(K(\alpha^{-1}(b))), c]_{\mathfrak{c}} \\
&\quad - [K^{-1}(K(\beta^{-1}(c))), \alpha^{-1}(b)]_{\mathfrak{g}} \\
&= -[b, \alpha \circ \beta^{-1}(c)]_{\mathfrak{b}} + [\beta \circ \alpha^{-1}(b), c]_{\mathfrak{c}} + [\alpha^{-1}(b), \beta^{-1}(c)]_{\mathfrak{g}} \\
&= -[b, \alpha \circ \beta^{-1}(c)]_{\mathfrak{b}} + [\alpha^{-1}(b), \beta^{-1}(c)]_{\mathfrak{g}} - [\beta \circ \alpha^{-1}(b), c]_{\mathfrak{c}^{\text{op}}} \\
&= -[b, \alpha \circ \beta^{-1}(c)]_T + [\alpha^{-1}(b), \beta^{-1}(c)]_T - [\beta \circ \alpha^{-1}(b), c]_T.
\end{aligned}$$

That is, the bracket of an element  $b \in \mathfrak{b}$  with an element  $c \in \mathfrak{c}$  is given by mapping  $b$  and/or  $c$  into each piece in turn and taking the bracket there. If the bracket is non-zero, it has a non-zero component in each piece.

We now consider the Lie coalgebra structure. We have that  $\mathfrak{g}$  is a sub-Lie bialgebra, so the cobracket on an element of  $\mathfrak{g}$  is simply  $\delta$ . For an element  $b \in \mathfrak{b} = \underline{\mathfrak{g}}$ , the unbraided cobracket structure of the double-bosonisation is

$$\begin{aligned}
\delta_T b &= \underline{\delta} b + r^{(2)} \otimes (r^{(1)} \triangleright b) - (r^{(1)} \triangleright b) \otimes r^{(2)} \\
&= \delta b + (r^{(1)} \triangleright b) \otimes \alpha(r^{(2)}) - \alpha(r^{(2)}) \otimes (r^{(1)} \triangleright b) \\
&\quad + r^{(2)} \otimes (r^{(1)} \triangleright b) - (r^{(1)} \triangleright b) \otimes r^{(2)} \\
&= \delta b + (r^{(1)} \triangleright b) \otimes (\alpha(r^{(2)}) - r^{(2)}) - (\alpha(r^{(2)}) - r^{(2)}) \otimes (r^{(1)} \triangleright b).
\end{aligned}$$

Since the bosonisation  $\mathfrak{g} \bowtie \mathfrak{c}^{\text{op}}$  is taken to be that of  $\mathfrak{c}^{\text{op}}$  in the category of  $\mathfrak{g}$ -modules with opposite infinitesimal braiding (see the proof of [Maj00, Theorem 3.10] for details), we have

$$\begin{aligned}
\delta_T c &= \underline{\delta} c + (r^{(2)} \triangleright c) \otimes r^{(1)} - r^{(1)} \otimes (r^{(2)} \triangleright c) \\
&= \delta c - (r^{(2)} \triangleright c) \otimes \beta(r^{(1)}) + \beta(r^{(1)}) \otimes (r^{(2)} \triangleright c) \\
&\quad + (r^{(2)} \triangleright c) \otimes r^{(1)} - r^{(1)} \otimes (r^{(2)} \triangleright c) \\
&= \delta c + (\beta(r^{(1)}) - r^{(1)}) \otimes (r^{(2)} \triangleright c) - (r^{(2)} \triangleright c) \otimes (\beta(r^{(1)}) - r^{(1)}). \quad \square
\end{aligned}$$

In what follows we will want to compute the bracket on general elements of  $T(\mathfrak{g})$  so we give this explicitly. Our notation for general elements will be as elements of the

direct sum vector space, usually writing  $b, g$  and  $c$  for elements of  $\mathfrak{b}, \mathfrak{g}$  and  $\mathfrak{c}$  respectively. We will also now suppress the isomorphisms  $\alpha, \beta, \dots$  when taking brackets.

**Theorem 5.1.2.** *Let  $\mathfrak{g}$  be a factorisable Lie bialgebra. The bracket  $[\ , \ ]_T$  on  $T(\mathfrak{g})$  is given as follows. For all  $b_1 \oplus g_1 \oplus c_1, b_2 \oplus g_2 \oplus c_2 \in T(\mathfrak{g})$  we have*

$$\begin{aligned} [b_1 \oplus g_1 \oplus c_1, b_2 \oplus g_2 \oplus c_2]_T = & ([b_1, b_2] + [b_1, g_2] - [b_1, c_2] + [g_1, b_2] - [c_1, b_2]) \\ & \oplus ([b_1, c_2] + [g_1, g_2] + [c_1, b_2]) \\ & \oplus ([c_1, b_2] + [c_1, g_2] - [c_1, c_2] + [g_1, c_2] + [b_1, c_2]) \end{aligned}$$

*Proof:* Immediate from the preceding lemma.  $\square$

## 5.2 The structure of $T(\mathfrak{g})$ , $\mathfrak{g}$ factorisable

We now investigate the structure of  $T(\mathfrak{g})$ , for  $\mathfrak{g}$  a factorisable Lie bialgebra. Our main results are Theorem 5.2.3 and Theorem 5.2.5. We see that these results are direct analogues of those for the Drinfel'd double, as recalled in Chapter 2.

### 5.2.1 The Lie algebra structure

We start by examining the Lie ideals of  $T(\mathfrak{g})$ .

**Lemma 5.2.1.** *The subspaces*

$$\begin{aligned} I_- &= \text{span}_k \{ x \oplus (-x) \oplus 0 \mid x \in \mathfrak{g} \} \\ I_0 &= \text{span}_k \{ x \oplus (-x) \oplus (-x) \mid x \in \mathfrak{g} \} \\ I_+ &= \text{span}_k \{ 0 \oplus x \oplus x \mid x \in \mathfrak{g} \} \end{aligned}$$

*are Lie subalgebras of  $T(\mathfrak{g})$ .*

*Proof:* we use the bracket on  $T(\mathfrak{g})$  as given in Theorem 5.1.2.

For  $I_-$ , let  $x \oplus (-x) \oplus 0, y \oplus (-y) \oplus 0 \in I_-$ . Then

$$\begin{aligned} [x \oplus (-x) \oplus 0, y \oplus (-y) \oplus 0]_T &= ([x, y] - [x, y] - [x, y]) \oplus ([x, y]) \oplus 0 \\ &= -[x, y] \oplus [x, y] \oplus 0 \\ &\in I_- \end{aligned}$$

Similarly, for  $I_0$ , let  $x \oplus (-x) \oplus (-x)$ ,  $y \oplus (-y) \oplus (-y) \in I_0$ . Then

$$\begin{aligned} [x \oplus (-x) \oplus (-x), y \oplus (-y) \oplus (-y)]_T &= [x, y] \oplus -[x, y] \oplus -[x, y] \\ &\in I_0 \end{aligned}$$

Finally, for  $I_+$ , let  $0 \oplus x \oplus x$ ,  $0 \oplus y \oplus y \in I_+$ . Then

$$\begin{aligned} [0 \oplus x \oplus x, 0 \oplus y \oplus y]_T &= 0 \oplus [x, y] \oplus [x, y] \\ &\in I_+ \end{aligned}$$

□

**Lemma 5.2.2.** *The subalgebras  $I_-$ ,  $I_0$  and  $I_+$  are ideals of  $T(\mathfrak{g})$ .*

*Proof:* let  $b \oplus g \oplus c \in T(\mathfrak{g})$ .

$I_-$ : let  $x \oplus (-x) \oplus 0 \in I_-$ . Then

$$\begin{aligned} [x \oplus (-x) \oplus 0, b \oplus g \oplus c]_T &= ([x, b] + [x, g] - [x, c] - [x, b]) \\ &\quad \oplus ([x, c] - [x, g]) \oplus 0 \\ &= [x, g - c] \oplus -[x, g - c] \oplus 0 \\ &\in I_- \end{aligned}$$

$I_0$ : let  $x \oplus (-x) \oplus (-x) \in I_0$ . Then

$$\begin{aligned} [x \oplus (-x) \oplus (-x), b \oplus g \oplus c]_T &= [x, b + g - c] \oplus -[x, b + g - c] \\ &\quad \oplus -[x, b + g - c] \\ &\in I_0 \end{aligned}$$

$I_+$ : let  $0 \oplus x \oplus x \in I_+$ . Then

$$\begin{aligned} [0 \oplus x \oplus x, b \oplus g \oplus c]_T &= 0 \oplus [x, b + g] \oplus [x, b + g] \\ &\in I_+ \end{aligned}$$

□

Combining these results, we obtain the full simplification of the Lie algebra structure of the triple, when  $\mathfrak{g}$  is factorisable.



**Theorem 5.2.3.**  $T(\mathfrak{g})$  is the direct sum of the ideals  $I_-$ ,  $I_0$  and  $I_+$ . Hence  $T(\mathfrak{g})$  is isomorphic to  $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$  as a Lie algebra.

*Proof:* we must show that the brackets between any two of  $I_-$ ,  $I_0$  and  $I_+$  are zero.

$[I_-, I_0]$ : let  $x \oplus (-x) \oplus 0 \in I_-$ ,  $y \oplus (-y) \oplus (-y) \in I_0$ . Then

$$\begin{aligned} [x \oplus (-x) \oplus 0, y \oplus (-y) \oplus (-y)]_T &= ([x, y] - [x, y] + [x, y] - [x, y]) \\ &\quad \oplus (-[x, y] + [x, y]) \oplus ([x, y] - [x, y]) \\ &= 0 \oplus 0 \oplus 0. \end{aligned}$$

Similarly,  $[I_+, I_0]$  and  $[I_+, I_-]$  are zero. Hence  $T(\mathfrak{g})$  is the direct sum of  $I_-$ ,  $I_0$  and  $I_+$ .

It is clear that  $I_-$ ,  $I_0$  and  $I_+$  are each isomorphic to  $\mathfrak{g}$  so we have the Lie algebra isomorphism  $T(\mathfrak{g}) \cong \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ . Notice, however, that the bracket on  $I_-$  is the opposite one (Lemma 5.2.1), so we can write

$$T(\mathfrak{g}) \cong I_+ \oplus I_0 \oplus I_- \cong \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}^{\text{op}},$$

which we recognise as the three ‘input’ Lie algebras of  $T(\mathfrak{g})$  with the bracket now diagonalised. Alternatively, we can write  $T(\mathfrak{g}) \cong \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$  using the usual isomorphism of a Lie algebra with its opposite.

Explicitly,

$$\begin{aligned} b \oplus g \oplus c &= \begin{pmatrix} 0 & \oplus & b+g & \oplus & b+g \end{pmatrix} \\ &\quad + \begin{pmatrix} b+g-c & \oplus & -b-g+c & \oplus & -b-g+c \end{pmatrix} \\ &\quad + \begin{pmatrix} -g+c & \oplus & g-c & \oplus & 0 \end{pmatrix} \end{aligned}$$

is the decomposition of a general element of  $T(\mathfrak{g})$  into a sum of elements in the ideals  $I_+$ ,  $I_0$  and  $I_-$  respectively. Then we have the two isomorphisms mentioned above.

$T(\mathfrak{g}) \cong \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}^{\text{op}}$ :

define  $\theta_1 : T(\mathfrak{g}) \rightarrow \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}^{\text{op}}$  by

$$b \oplus g \oplus c \mapsto (b+g) \oplus (b+g-c) \oplus (-g+c)$$

$T(\mathfrak{g}) \cong \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ :

define  $\theta_2 : T(\mathfrak{g}) \rightarrow \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$  by

$$b \oplus g \oplus c \mapsto (b + g) \oplus (b + g - c) \oplus (g - c)$$

It is easily checked that these are Lie algebra isomorphisms.  $\square$

We have an immediate corollary.

**Corollary 5.2.4.** *The rank of  $T(\mathfrak{g})$  is three times that of  $\mathfrak{g}$ .*  $\square$

This may also be proved independently of the above theorem by examining the possible Abelian subalgebras of  $T(\mathfrak{g})$ . Indeed, we can use the above isomorphism to see that the Cartan subalgebra of  $T(\mathfrak{g})$  is the direct sum of the three incarnations of the Cartan subalgebra of  $\mathfrak{g}$ .

### 5.2.2 The Lie coalgebra structure

Recall that the Lie coalgebra structure of a Lie bialgebra is completely determined by the Lie algebra and the quasitriangular structure,  $r$ . From the previous section we have a Lie algebra isomorphism of  $T(\mathfrak{g})$  with  $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ . Furthermore, double-bosonisation comes with an explicit expression for its quasitriangular structure. This is given in Proposition 2.2.3. We will identify the image of the quasitriangular structure under the Lie algebra isomorphism and so express  $T(\mathfrak{g})$  as a twisting by a cocycle of the direct sum structure.

**Theorem 5.2.5.** *Let  $\mathfrak{g}$  be a factorisable Lie bialgebra. As a Lie bialgebra,  $T(\mathfrak{g})$  is isomorphic to  $(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g})_\chi$ , the twisting by*

$$\chi = r_{AB} - \tau(r_{AB}) + r_{BC} - \tau(r_{BC}) + r_{AC} - \tau(r_{AC})$$

*of the direct sum coalgebra structure where we take  $r \oplus -r_{21} \oplus r$  as the quasitriangular structure on the direct sum Lie algebra.*

*Here, for example,  $r_{AB} = (r^{(1)} \oplus 0 \oplus 0) \otimes (0 \oplus r^{(2)} \oplus 0)$ .*

*Proof:* Recall the definition of the (Lie algebra) isomorphism  $\theta_2$  above, that is,

$$\theta_2 : T(\mathfrak{g}) \rightarrow \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}, \quad b \oplus g \oplus c \mapsto (b + g) \oplus (b + g - c) \oplus (g - c).$$

We will write  $r_T$  for the quasitriangular structure on  $T(\mathfrak{g})$ . From the proposition, we have  $r_T = 0 \oplus r \oplus 0 + (0 \oplus 0 \oplus f^a) \otimes (e_a \oplus 0 \oplus 0)$  in the direct sum notation, with summation over  $a$  understood. Here  $\{e_a\}$  is a basis of  $\underline{\mathfrak{g}}$  and  $\{f^a\}$  is a dual basis. Hence

$$\begin{aligned} (\theta_2 \otimes \theta_2)(r_T) &= \theta_2(0 \oplus r^{(1)} \oplus 0) \otimes \theta_2(0 \oplus r^{(2)} \oplus 0) \\ &\quad + \theta_2(0 \oplus 0 \oplus f^a) \otimes \theta_2(e_a \oplus 0 \oplus 0) \\ &= (r^{(1)} \oplus r^{(1)} \oplus r^{(1)}) \otimes (r^{(2)} \oplus r^{(2)} \oplus r^{(2)}) \\ &\quad + (0 \oplus -f^a \oplus -f^a) \otimes (e_a \oplus e_a \oplus 0) \end{aligned}$$

This expression may be simplified as follows. Label the three copies of  $\mathfrak{g}$  in the direct sum as  $\mathfrak{g}_A$ ,  $\mathfrak{g}_B$  and  $\mathfrak{g}_C$  and for elements of the tensor product of any two of these, write the appropriate subscripts. For example, we will write  $a_{AB}$  for  $(a_{(1)} \oplus 0 \oplus 0) \otimes (0 \oplus a_{(2)} \oplus 0)$  or  $(a^{(1)} \oplus 0 \oplus 0) \otimes (0 \oplus a^{(2)} \oplus 0)$  (we will use subscripts irrespective of whether we are writing upper or lower Sweedler indices).

We observe that  $f^a \otimes e_a$  is precisely the inverse Killing form—or in its quasitriangular form,  $2r_+$ , the symmetric part of  $r$ . Hence expanding out the tensor products and rewriting in our subscript notation, we have the following:

$$\begin{aligned} (\theta_2 \otimes \theta_2)(r_T) &= r_{AA} + r_{AB} + r_{AC} + r_{BA} + r_{BB} + r_{BC} + r_{CA} + r_{CB} + r_{CC} \\ &\quad - (f^a \otimes e_a)_{BA} - (f^a \otimes e_a)_{BB} - (f^a \otimes e_a)_{CA} - (f^a \otimes e_a)_{CB} \end{aligned}$$

However,  $f^a \otimes e_a = 2r_+ = r + \tau(r)$  so

$$\begin{aligned} (f^a \otimes e_a)_{BA} &= (r + \tau(r))_{BA} \\ &= r^{(1)}_B \otimes r^{(2)}_A + r^{(2)}_B \otimes r^{(1)}_A \\ &= r_{BA} + \tau(r_{AB}) \end{aligned}$$

and similarly for the other terms.

Hence we obtain

$$\begin{aligned}
 (\theta_2 \otimes \theta_2)(r_T) &= r_{AA} + r_{AB} + r_{AC} + r_{BA} + r_{BB} + r_{BC} + r_{CA} + r_{CB} + r_{CC} \\
 &\quad - (r_{BA} + \tau(r_{AB})) - (r_{BB} + \tau(r_{BB})) \\
 &\quad - (r_{CA} + \tau(r_{AC})) - (r_{CB} + \tau(r_{BC})) \\
 &= r_{AA} - \tau(r_{BB}) + r_{CC} \\
 &\quad + (r_{AB} - \tau(r_{AB})) + (r_{BC} - \tau(r_{BC})) + (r_{AC} - \tau(r_{AC})). \tag{5.1}
 \end{aligned}$$

Notice first that  $r_\oplus = r_{AA} - \tau(r_{BB}) + r_{CC}$  is the direct sum quasitriangular structure on  $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ , choosing the opposite quasitriangular structure for the central copy of  $\mathfrak{g}$ . Now set

$$\chi = r_T - r_\oplus = r_{AB} - \tau(r_{AB}) + r_{BC} - \tau(r_{BC}) + r_{AC} - \tau(r_{AC}).$$

Then  $\chi + \chi_{21} = 0$ , as is easily seen. That is,  $\chi$  is symmetric and we need only check the identity  $(\text{id} \otimes \delta_\oplus)\chi + \text{cyclic} + \llbracket \chi, \chi \rrbracket = 0$  to see that  $\chi$  satisfies the conditions for a cocycle and hence defines a twisting of the direct sum. This identity follows immediately, however, since we can consider  $\chi$  as a sum  $\chi = \chi_{AB} + \chi_{BC} + \chi_{AC}$ , where  $\chi_{AB} = r_{AB} - \tau(r_{AB})$  and similarly for the others. It is known from the proof that the Drinfel'd double is a twisting of a direct sum (see [Maj95, Theorem 8.2.5]) that terms of precisely the form  $\chi_{AB}$ , etc., satisfy the required identity.  $\square$

### 5.3 Relationship with the Drinfel'd double

The above description of the triple in Theorem 5.2.5 is clearly reminiscent of that for the Drinfel'd double (Theorem 2.2.1). More than that, we expect at least one copy of the double to sit inside the triple. For example, in the bosonisation picture we have  $D(\mathfrak{g}) \cong \mathfrak{g} \bowtie \mathfrak{g}^{*\text{op}}$  and, of course, we defined the triple as  $T(\mathfrak{g}) = \mathfrak{g} \bowtie \mathfrak{g} \bowtie \mathfrak{g}^{*\text{op}}$ . Below, we expand these ideas.

Firstly, let  $\mathfrak{g}$  be a quasitriangular Lie bialgebra, not necessarily factorisable. We observe that we can write the triple as a matched pair of Lie algebras, as in [Maj90a]. Since we have a Lie algebra structure on the triple, we can break this up as the Lie

algebras  $\underline{\mathfrak{g}}$  and  $\mathfrak{g} \bowtie \underline{\mathfrak{g}}^{*\text{op}}$  and make a matched pair by actions between them. Identifying the coalgebra structure on this matched pair, we can rewrite the bialgebra structure of  $T(\mathfrak{g})$  as follows.

**Theorem 5.3.1.** *Let  $\mathfrak{g}$  be a quasitriangular Lie bialgebra. Then  $T(\mathfrak{g})$  is a double cross sum Lie algebra and a semidirect Lie coalgebra, written*

$$T(\mathfrak{g}) = \underline{\mathfrak{g}} \bowtie (\mathfrak{g} \bowtie \underline{\mathfrak{g}}^{*\text{op}}).$$

*Proof:* We have a left action of  $\mathfrak{g} \bowtie \underline{\mathfrak{g}}^{*\text{op}}$  on  $\underline{\mathfrak{g}}$ ,

$$\begin{aligned} \alpha : (\mathfrak{g} \bowtie \underline{\mathfrak{g}}^{*\text{op}}) \otimes \underline{\mathfrak{g}} &\rightarrow \underline{\mathfrak{g}} \\ \alpha((g \oplus c) \otimes b) &= \text{ad}_g(b) - \underline{b}_{(1)} \langle \underline{b}_{(2)}, c \rangle \end{aligned}$$

and a right action of  $\underline{\mathfrak{g}}$  on  $\mathfrak{g} \bowtie \underline{\mathfrak{g}}^{*\text{op}}$ ,

$$\begin{aligned} \beta : (\mathfrak{g} \bowtie \underline{\mathfrak{g}}^{*\text{op}}) \otimes \underline{\mathfrak{g}} &\rightarrow \mathfrak{g} \bowtie \underline{\mathfrak{g}}^{*\text{op}} \\ \beta((g \oplus c) \otimes b) &= -\underline{c}_{(1)} \langle \underline{c}_{(2)}, b \rangle - 2r_+^{(1)} \langle c, r_+^{(2)} \triangleright b \rangle \end{aligned}$$

with  $\triangleright$  being the adjoint action. Note that these are exactly the terms in the bracket defined on  $T(\mathfrak{g})$  between these two pieces, so we have a matched pair and the Lie bracket on the double cross sum coming from this is exactly that of  $T(\mathfrak{g})$ .

We notice that  $\underline{\mathfrak{g}} \bowtie \mathfrak{g}$  occurs as a sub-Lie bialgebra of  $T(\mathfrak{g})$  and in particular that elements of  $\underline{\mathfrak{g}}^{*\text{op}}$  do not appear in the cobracket on elements of  $\underline{\mathfrak{g}}$ . This cobracket is the one obtained by bosonisation, which is by definition a semidirect coalgebra by a Lie coaction. For  $\underline{\mathfrak{g}} \bowtie \mathfrak{g}$  the Lie coaction of  $\mathfrak{g}$  on  $\underline{\mathfrak{g}}$  is  $\gamma : \underline{\mathfrak{g}} \rightarrow \mathfrak{g} \otimes \underline{\mathfrak{g}}$ ,  $\gamma(b) = r^{(2)} \otimes r^{(1)} \triangleright b$  for  $b \in \underline{\mathfrak{g}}$ . Then we can extend this coaction to one of  $\mathfrak{g} \bowtie \underline{\mathfrak{g}}^{*\text{op}}$  on  $\underline{\mathfrak{g}}$  by letting  $\underline{\mathfrak{g}}^{*\text{op}}$  coact by zero. So we can write  $T(\mathfrak{g}) = \underline{\mathfrak{g}} \bowtie (\mathfrak{g} \bowtie \underline{\mathfrak{g}}^{*\text{op}})$  as coalgebras.  $\square$

We can use the isomorphism of  $\mathfrak{g} \bowtie \underline{\mathfrak{g}}^{*\text{op}}$  with  $D(\mathfrak{g})$  to give a version of this theorem involving the double which is independent of any particular realisation of the double. The isomorphism is explicitly given by

$$\sigma : D(\mathfrak{g}) \rightarrow \mathfrak{g} \bowtie \underline{\mathfrak{g}}^{*\text{op}}, \quad \sigma(h \oplus d) = (h - r^{(1)} \langle r^{(2)}, d \rangle) \oplus d$$

for  $h \in \mathfrak{g}$ ,  $d \in \mathfrak{g}^{*\text{op}}$  and with  $D(\mathfrak{g}) = \mathfrak{g} \bowtie \mathfrak{g}^{*\text{op}}$ . Note that the inverse is

$$\sigma^{-1} : \mathfrak{g} \bowtie \mathfrak{g}^{*\text{op}} \rightarrow D(\mathfrak{g}), \quad \sigma^{-1}(g \oplus c) = (g + r^{(1)} \langle r^{(2)}, c \rangle) \oplus c$$

for  $g \in \mathfrak{g}$ ,  $c \in \mathfrak{g}^{*\text{op}}$ . That  $\sigma$  is a bialgebra isomorphism may be checked along the same lines as the proof in [Maj00, Example 3.9] that  $D(\mathfrak{g}) \cong \underline{\mathfrak{g}}^* \bowtie \mathfrak{g}$  (i.e. using the opposite conventions).

**Corollary 5.3.2.** *Let  $\mathfrak{g}$  be a quasitriangular Lie bialgebra. Then  $T(\mathfrak{g})$  is isomorphic to a double cross sum Lie algebra and a semidirect Lie coalgebra*

$$T(\mathfrak{g}) \cong \underline{\mathfrak{g}} \bowtie D(\mathfrak{g}).$$

*Proof:* Define the induced actions

$$\hat{\alpha} : D(\mathfrak{g}) \otimes \underline{\mathfrak{g}} \rightarrow \underline{\mathfrak{g}}, \quad \hat{\alpha} = \alpha \circ (\sigma \otimes \text{id})$$

and

$$\hat{\beta} : D(\mathfrak{g}) \otimes \underline{\mathfrak{g}} \rightarrow D(\mathfrak{g}), \quad \hat{\beta} = \sigma^{-1} \circ \beta \circ (\sigma \otimes \text{id}).$$

Explicit expressions may be obtained from Theorem 5.3.1. These actions give a matched pair  $(\underline{\mathfrak{g}}, D(\mathfrak{g}))$ . For the coalgebra, we let  $\mathfrak{g} \subset D(\mathfrak{g})$  coact by  $\gamma$  on  $\underline{\mathfrak{g}}$  as above and let  $\mathfrak{g}^{*\text{op}}$  coact by zero.  $\square$

We now restrict to  $\mathfrak{g}$  factorisable. Recall the description above of  $T(\mathfrak{g})$  as a direct sum Lie algebra with twisted coalgebra structure. Recall also the similar description of  $D(\mathfrak{g})$ , as in Theorem 2.2.1. Notice that if we take only the terms in equation (5.1) (on page 107) involving the  $A$  and  $B$  copies, we have  $r_{AB} - \tau(r_{BB}) + r_{AB} - \tau(r_{AB})$  which is precisely the quasitriangular structure on the double  $D(\mathfrak{g})$  in the form given by that theorem. So we can describe the triple in terms of the double as follows.

**Corollary 5.3.3.** *As Lie bialgebras,  $T(\mathfrak{g}) \cong D(\mathfrak{g}) \bowtie \mathfrak{g}$*

*Proof:* We see from the proof of Theorem 5.2.5 and the preceding comments that we have the bialgebra isomorphism  $T(\mathfrak{g}) \cong (\mathfrak{g} \bowtie \mathfrak{g}) \bowtie \mathfrak{g}$ . Here, the first double cross cosum is as described in Theorem 2.2.1 and the twisting in the second is by

$$\chi_{AC} + \chi_{BC} = (r_{AC} - \tau(r_{AC})) + (r_{BC} - \tau(r_{BC}))$$

which is a suitable cocycle, as before. Then the isomorphism of  $\mathfrak{g} \bowtie \mathfrak{g}$  with  $D(\mathfrak{g})$  for  $\mathfrak{g}$  factorisable gives the result.  $\square$

#### 5.4 Real forms and half-real forms of the triple

We now work over  $\mathbb{C}$  and consider real and half-real forms of the triple. We consider only the factorisable case. Recall that a real form of a complex Lie algebra  $\mathfrak{g}$  is a choice of basis for  $\mathfrak{g}$  such that all structure constants are real. There are two particularly natural real forms, namely the split and compact forms, but other non-isomorphic forms too. See, for example, [FH91] for more on real forms. The natural basis for a split form gives us a bialgebra over  $\mathbb{R}$  and as the results of the preceding sections hold over any field of characteristic not 2 this case is dealt with.

We define a half-real form to be a choice of basis with real Lie algebra structure constants and imaginary Lie coalgebra structure constants. There will, of course, generally be non-isomorphic half-real forms of the same Lie bialgebra. This concept has been introduced in [Maj90a], as useful in describing Iwasawa decompositions. Half-real forms are equivalent to real forms but here we will again find them a more useful language. In particular, the natural choice of basis for compact forms of simple Lie algebras leads us to consider half-real forms. However, note that a half-real form  $(\mathfrak{u}, r)$  is a complex Lie bialgebra, not a bialgebra over  $\mathbb{R}$ . This is because the quasitriangular structure  $r$  involves  $i$ , so  $\mathfrak{u}$  is not quasitriangular over  $\mathbb{R}$ . When  $\mathfrak{g}$  is quasitriangular, we say  $r$  is of real type if the symmetric part of  $r$ ,  $2r_+$ , is in fact real in the basis for  $\mathfrak{u}$ .

**Lemma 5.4.1.** *Let  $\mathfrak{g}$  be a complex factorisable Lie bialgebra. Let  $(\mathfrak{u}, r)$  be a half-real form of  $\mathfrak{g}$  with  $r$  of real type. Then the transmutation of  $\mathfrak{u}$ ,  $\underline{\mathfrak{u}}$ , as described in Definition 2.2.3 is a real-real form of the transmutation  $\underline{\mathfrak{g}}$ . That is,  $\underline{\mathfrak{u}}$  has real bracket and cobracket structure constants. Note that  $\underline{\mathfrak{u}}$  is therefore self-dual.*

*Proof:* We are in the case of  $r$  real type, i.e.  $2r_+$  real. Then since by definition  $\mathfrak{u}$  has real bracket structure constants, the braided-Lie cobracket structure constants are real, for we recall that

$$\underline{\delta}x = 2r_+^{(1)} \otimes [x, r_+^{(2)}]$$

for all  $x \in \mathfrak{u}$ . Self-duality is ensured by [Maj00, Example 3.3].  $\square$

Therefore we have the following theorem.

**Theorem 5.4.2.** *Let  $\mathfrak{g}$  be a factorisable Lie bialgebra over  $\mathbb{C}$ . Let  $(\mathfrak{u}, r)$  be a half-real form of  $\mathfrak{g}$  with real type quasitriangular structure. Considering  $\underline{\mathfrak{u}}$  as in the preceding lemma as a complex braided-Lie bialgebra, define the triple of  $\mathfrak{u}$  to be the double-bosonisation  $T(\mathfrak{u}) = \underline{\mathfrak{u}} \bowtie \mathfrak{u} \ltimes \underline{\mathfrak{u}}^{\text{op}}$ .*

*Then  $T(\mathfrak{u})$  is a half-real form of  $T(\mathfrak{g})$  with real type quasitriangular structure.*

*Proof:* The brackets on  $\mathfrak{u}$  and  $\underline{\mathfrak{u}}$  are real by assumption, as are the braided-Lie cobracket  $\underline{\delta}$  and the symmetric part  $2r_+$  of the quasitriangular structure on  $\mathfrak{u}$ . Examining the definitions of the brackets in the triple, we see that these then define a real bracket on  $T(\mathfrak{u})$ .

The quasitriangular structure  $r_T$  is not real since  $r$  on  $\mathfrak{u}$  is not real. The induced quasitriangular structure on the triple is recalled in Proposition 2.2.3 and may be written as  $r_T = 0 \oplus r \oplus 0 + (0 \oplus 0 \oplus f^a) \otimes (e_a \oplus 0 \oplus 0)$  where  $\{e_a\}$  is a basis of  $\underline{\mathfrak{u}}$  and  $\{f^a\}$  is a dual basis. Note that the part  $(0 \oplus 0 \oplus f^a) \otimes (e_a \oplus 0 \oplus 0)$  is real. The dual pairing we are using is the Killing form which is real since  $\underline{\mathfrak{u}}$  is real as a Lie algebra. Hence the symmetric part  $2(r_T)_+$  is real:  $2r_+$  is real and any contribution from  $f^a \otimes e_a$  can only be real. So  $r_T$  is of real type.  $\square$

Conversely, if  $r$  is not of real type then the bracket on the triple is not real, since  $\underline{\delta}$  and  $2r_+$  are not real. Then  $T(\mathfrak{u}) = \underline{\mathfrak{u}} \bowtie \mathfrak{u} \ltimes \underline{\mathfrak{u}}^{\text{op}}$  is not a half-real form of  $T(\mathfrak{g})$ .

## 5.5 The triangular case

Recall that a quasitriangular Lie bialgebra  $(\mathfrak{g}, r)$  is said to be triangular if  $r$  has zero symmetric part. Then if  $\mathfrak{b}$  is a  $\mathfrak{g}$ -module the associated infinitesimal braiding is also zero: we have

$$\psi(a \otimes b) = 2r_+ \triangleright (a \otimes b - b \otimes a) = 0.$$



So a braided-Lie bialgebra  $(\mathfrak{b}, [\ , \ ]_{\mathfrak{b}}, \underline{\delta})$  in the category  ${}_{\mathfrak{g}}\mathcal{M}$  is a  $\mathfrak{g}$ -covariant bialgebra with  $d\underline{\delta} = 0$ . In particular, we see that this last condition means  $\mathfrak{b}$  is a  $\mathfrak{g}$ -covariant unbraided Lie bialgebra in the category.

We consider the adjoint representation of  $\mathfrak{g}$  in  ${}_{\mathfrak{g}}\mathcal{M}$ . The transmutation  $\underline{\mathfrak{g}}$  as defined in Definition 2.2.3 has the adjoint module structure and the Lie bracket of  $\mathfrak{g}$  but the zero braided-Lie cobracket, since  $\mathfrak{g}$  is triangular. Moreover, this is essentially forced upon us.

**Lemma 5.5.1.** *Let  $(\mathfrak{g}, r)$  be a non-Abelian triangular Lie bialgebra and let  $\underline{\mathfrak{g}} \in {}_{\mathfrak{g}}\mathcal{M}$  be the adjoint representation of  $\mathfrak{g}$ , made a Lie algebra in the category by the Lie algebra of  $\mathfrak{g}$ . Let  $\underline{\delta}$  be a  $\mathfrak{g}$ -covariant cobracket on  $\underline{\mathfrak{g}}$ . Then  $\underline{\delta} = 0$ .*

*Proof:* We have  $\mathfrak{g}$ -covariance of the coalgebra structure in the form

$$\underline{\delta}(\xi \triangleright x) = \xi \triangleright \underline{\delta}x$$

for  $\xi \in \mathfrak{g}$ ,  $x \in \underline{\mathfrak{g}}$ . We also have the zero coboundary property for  $\underline{\delta}$  as described above. Explicitly, for  $a, b \in \underline{\mathfrak{g}}$

$$\underline{\delta}([a, b]) = \underline{\text{ad}}_a(\underline{\delta}b) - \underline{\text{ad}}_b(\underline{\delta}a). \quad (5.2)$$

Here,  $\underline{\text{ad}}$  refers to the adjoint action of  $\underline{\mathfrak{g}}$  on itself, i.e. in the category  ${}_{\underline{\mathfrak{g}}}\mathcal{M}$ , or its extension to  $\underline{\mathfrak{g}} \otimes \underline{\mathfrak{g}}$ . Define an isomorphism of Lie algebras  $\iota : \mathfrak{g} \rightarrow \underline{\mathfrak{g}}$  by  $\iota = \text{id}$ , the identity map. We can now write  $\triangleright$ , the adjoint action of  $\mathfrak{g}$  on  $\underline{\mathfrak{g}}$ , as  $\triangleright : \mathfrak{g} \otimes \underline{\mathfrak{g}} \rightarrow \underline{\mathfrak{g}}$ ,  $\triangleright(\xi \otimes x) = \underline{\text{ad}}_{\iota(\xi)}(x)$ , or equivalently,  $\triangleright = \underline{\text{ad}} \circ (\iota \otimes \text{id})$ .

So, setting  $a = \iota(\xi)$  and  $b = x$  in (5.2) and writing in terms of  $\underline{\text{ad}}$ , we have the following two equalities:

$$\begin{aligned} \underline{\delta}(\underline{\text{ad}}_{\iota(\xi)}(x)) &= \underline{\text{ad}}_{\iota(\xi)}(\underline{\delta}x) \\ \underline{\delta}(\underline{\text{ad}}_{\iota(\xi)}(x)) &= \underline{\text{ad}}_{\iota(\xi)}(\underline{\delta}x) - \underline{\text{ad}}_x(\underline{\delta}(\iota(\xi))) \end{aligned}$$

Now we see that we have  $\underline{\text{ad}}_x(\underline{\delta}(\iota(\xi))) = 0$  and our choices of  $\iota(\xi), x \in \underline{\mathfrak{g}}$  were arbitrary. So we conclude that if  $\mathfrak{g}$  is not Abelian, so that  $\underline{\text{ad}}$  and  $\iota$  are not identically zero, we must have  $\underline{\delta} = 0$ .  $\square$

We now assume  $\mathfrak{g}$  is not Abelian. Notice that  $\underline{\mathfrak{g}}$  is now not self-dual, as it has a non-zero bracket and zero braided-cobracket. The dual  $\underline{\mathfrak{g}}^*$  will have zero bracket and non-zero braided-cobracket, namely the (unbraided) Kirillov–Kostant cobracket. We recall the definition of the triple, using the simplifications we have deduced above.

**Definition 5.5.1.** *Let  $\mathfrak{g}$  be a non-Abelian, finite-dimensional, triangular Lie bialgebra over a field  $k$  of characteristic not 2. Consider the transmutation  $\underline{\mathfrak{g}}$  as described above. Define  $T(\mathfrak{g})$  to be the double-bosonisation  $\underline{\mathfrak{g}} \bowtie_{\text{ad}} \mathfrak{g} \bowtie_{\text{coad}} \underline{\mathfrak{g}}^*$ .*

We have dispensed with the opposite bracket on the dual, as the bracket is zero there.

Examining the bracket from the double-bosonisation, we can write the Lie algebra structure of the triple in this case as follows.

**Proposition 5.5.2.** *Let  $\mathfrak{g}$  be a triangular Lie bialgebra and  $T(\mathfrak{g})$  the triple as defined above. Then we have the Lie algebra isomorphism*

$$T(\mathfrak{g}) \cong (\mathfrak{g} \bowtie_{\text{ad}} \mathfrak{g})_{\text{coad}} \bowtie \mathfrak{g}^*.$$

Here  $\text{ad}$  and  $\text{coad}$  refer to the adjoint and coadjoint actions, respectively, and both parts of  $\mathfrak{g} \bowtie_{\text{ad}} \mathfrak{g}$  act on  $\mathfrak{g}^*$  by the coadjoint action.

*Proof:* This follows immediately from examining the brackets in the double-bosonisation and identifying the non-zero parts. In particular, we note that

$$2r_+^{(1)} \langle \varphi, r_+^{(2)} \triangleright x \rangle = -x_{(1)} \langle \varphi, \underline{(2)} \rangle = 0$$

since  $\delta x = 0$ . We also use  $\delta \varphi = \delta \varphi$ . Since  $\underline{\mathfrak{g}}$  has the same Lie algebra as  $\mathfrak{g}$ , on dualisation  $\underline{\mathfrak{g}}^*$  has the same Lie coalgebra as  $\mathfrak{g}^*$ .  $\square$

It does not appear that any further simplification of the description of the triple in the triangular case is possible.

## Chapter 6

# Conclusion

We end by making some remarks on aspects of this work which remain unresolved and suggest some possible further directions for future work.

### 6.1 Lie induction

As we have described in the introduction, one motivation for us to use Lie induction to provide insight into the simple Lie (bi-)algebras. There are some outstanding questions prompted even by the small number of examples we have given.

We have described an algorithm for calculating candidates for inductions in Section 3.2.3. However, we have seen that there are obstructions to the existence of such candidates and also the question of ‘flatness’. By flatness, we mean that taking different routes should yield the same induction.

Even in the case where we obtain a sensible candidate braided-Lie bialgebra, we have a further practical problem, namely deciding whether the double-bosonisation is semisimple, let alone simple. Our work on the triple in Chapter 5 illustrates this general problem when working with double-bosonisation, in a different context. For Lie induction, we have the following questions:

- i)* How do we fix the remaining values in the Cartan matrix for the induced algebra  $\mathfrak{g}$ ?  
Note that we have the sub-matrix corresponding to  $\mathfrak{g}_0$  and the row corresponding to the new simple root from the highest weight of  $\mathfrak{b}_{-1}$ .
- ii)* Where, algebraically, do the properties of the Cartan matrix come from, for example,  $C_{ij} = 0 \iff C_{ji} = 0$ ? The restriction on the values in the Cartan matrix

is related to the restriction on the number of non-zero graded parts we may have in  $\mathfrak{b}$ —where do these come from?

- iii) Is there a general easy test to decide if a double-bosonisation  $\mathfrak{b} \bowtie \tilde{\mathfrak{g}}_0 \bowtie \mathfrak{b}^{*\text{op}}$  is simple?

These are clearly not independent: an answer to the first two questions would give us an effective answer to the third.

However, the results and examples we have, though partial, give some hints as to other settings where these obstructions might not occur. We have some candidates for a finite-dimensional  $E_9$ ,  $F_5$  and  $G_3$ , which must certainly not be simple. This is in some ways encouraging. Had we found that no modules were possible candidates for any induction, there would be little more to say. It is plausible, though, that in an alternative setting—for example that of quasi-Lie algebras—the non-simple candidates we have might in fact be simple. Even when no modules exist in the Lie setting due to dimension restrictions, an alternative setting might provide these. To say more we would need to see how far the theory we have developed carried through and in particular we would need a classification result in place of Theorem 3.2.4. Of course, this theorem supposed the knowledge of the classification of the simples in the first place, although a case-by-case inductive approach may be possible.

We have built here on the work begun in [Maj00] and many of the comments concluding that paper apply equally well here. We have also restricted ourselves to working over the complex field and to considering the standard quasitriangular structure. As noted there, it ought to be possible consider twisting and  $*$ -structures on the braided-Lie bialgebras and we now see that they would have to be compatible with the graded structure.

We also recall that the double-bosonisation construction can be defined when working over fields of any characteristic except two. The theorem of Azad, Barry and Seitz ([ABS90]) we use in Section 3.1.1 holds except for the following algebra-characteristic pairs, called *special*:  $(B_l, \text{char } K = 2)$ ,  $(C_l, 2)$ ,  $(F_4, 2)$ ,  $(G_2, 2)$  and  $(G_2, 3)$ . So we expect that the inductive method ought to carry over to (most) positive characteristics. This,

and an analysis of these special pairs, would be an interesting direction for further work.

It seems natural to extend our field of view to Kac-Moody Lie algebras in general, since we have dealt here with the finite-dimensional case only. The definitions of a braided-Lie bialgebra and of double-bosonisation do not need finite-dimensionality: the only result we use that does is the quasitriangularity of the double-bosonisation but with care this should not be a problem. We may require formal power series, for example, or an alternative formulation, such as the semi-classical version of a weak quasitriangular structure.

## 6.2 Quantum Lie induction

The principal aim of Chapter 4 was to extend the work of Majid ([Maj00], [Maj99]) and Chapter 3 to the quantum setting. We have seen that there is a strong correspondence between Lie induction and its quantum counterpart, in that many structural features appear in both. However, there are some differences. On the plus side, our results in the quantum setting are not restricted to finite-type root data, corresponding to the finite-dimensional setting of Chapter 3. We have also considered the deletion of an arbitrary number of nodes from the associated Dynkin diagram, including the deletion of all nodes (the case explicitly described in [Maj02, Chapter 18]).

However, we currently have less complete knowledge of the structure of the associated braided groups  $B = B(\mathfrak{T}, \mathfrak{J}, \iota)$ . We have a set of generators for  $B$  (Theorem 4.5.1) but not a presentation: we need a description of the relations in this algebra. It is clear that  $B$  should inherit some of the  $q$ -Serre relations—relation (R6) in Definition 2.3.10 for  $i, j \in I \setminus J$ . It is not completely clear that we should have no further relations, though.

Knowledge of a presentation for  $B$  should also allow us to complete our description of the coalgebra structure. We conjecture that the space of braided-primitive elements of  $B$  should have the same dimension as  $\mathfrak{b}$ , the braided-Lie bialgebra associated to the same deletion/sub-root datum, and that the relations in  $B$  should come from the Lie algebra structure on  $\mathfrak{b}$ . Then we would interpret  $B$  as a braided enveloping algebra of

$\mathfrak{b}$ ,  $\underline{U}_q(\mathfrak{b})$  say, which would make sense of the identity

$$U_q(\mathfrak{b} \rtimes \tilde{\mathfrak{g}}_0 \ltimes \mathfrak{b}^{*\text{op}}) \equiv U_q(\mathfrak{g}) \equiv \underline{U}_q(\mathfrak{b}) \rtimes \widetilde{U_q(\mathfrak{g}_0)} \ltimes \underline{U}_q(\mathfrak{b}^{*\text{op}}).$$

Such an expression would indicate that Lie induction behaves very well with respect to quantisation and would considerably strengthen the argument for studying it further.

Beyond the questions which parallel those from the semi-classical theory, we have some which exist only in the quantum setting. One area for future work would be to attempt to identify quantum groups obtained by this induction that do not have classical (i.e.  $q \rightarrow 1$ ) counterparts—purely quantum phenomena. It is not clear even whether such objects of this type exist, though.

It is the author's opinion that the ideas of Lie induction should not be thought of as simply an alternative description of some, albeit important, algebraic structures, but in fact a potential method of proof. We have considered one aspect of this—the classification problem—in Section 3.2 and discussed it further above.

For the quantum case, we have a different type of proof method in mind, namely a genuine inductive method. For example, one may consider the existence of the Poincaré–Birkoff–Witt-type (PBW-type) basis and the Lusztig–Kashiwara canonical (or crystal) basis on  $U_q(\mathfrak{T})$ . For an induction, the base case is rank one root data, for which the existence should be provable directly (and hopefully, easily). Then for the inductive step, one should show that such a basis is induced on  $B$ . If  $B$  is indeed  $\underline{U}_q(\mathfrak{b})$  as above, then this is plausible.

In this direction, Ufer ([Ufe04]) has shown that one obtains PBW-type bases on the Nichols algebra of certain finite-dimensional  $U_q(\mathfrak{g})$ -modules, via a braiding coming from the quasi- $\mathcal{R}$ -matrix. The Nichols algebra is a certain quotient of the tensor algebra: our previous conjecture on the structure of  $B$  could be rephrased to say that  $B$  is a Nichols algebra generated by the braided-primitive elements (which should correspond to  $\mathfrak{b}$ ), or possibly a quotient of this. We would like to extend Ufer's results to all modules arising from Lie induction, not just finite-dimensional ones.

The ultimate aim of such an approach would be to obtain a deeper understanding of these bases and their properties. In principle, one stands a better chance of understand-

ing them on the ‘smaller’ object  $B$ . It may be possible to see that the module properties of  $B$  force conditions on, for example, the corresponding structure constants. Even if full proofs are not immediately forthcoming, performing explicit calculations may be simplified. The package QuaGroup ([Qua]) for the computer program GAP ([GAP]) works with the PBW-type basis on  $U_q(\mathfrak{g})$  (for  $\mathfrak{g}$  of small rank) and we have begun to develop some code to calculate examples of deletions in GAP.

### 6.3 The triple construction

All our results from Chapter 5 on the triple should be the semi-classical version of, and provide insight into, quantum group versions of similar constructions. The analogous general double-bosonisation is of course known (Theorem 2.3.9) and for  $H$  finite-dimensional the special case which we would call

$$T(H) \stackrel{\text{def}}{=} \underline{H} \bowtie H \bowtie \underline{H}^{*\text{op}}$$

is again a canonical example (using quantum group (co)adjoint actions). Its particular structure has not been studied but it is quasitriangular (from the general theory) at least in the finite-dimensional case, and is an extension of the Drinfel’d quantum double  $D(H)$  (Definition 2.3.5). In terms of applications, the triple  $T(H)$  can be expected to extend the role of the double  $D(H)$ . For example, in non-commutative differential geometry the bicovariant differential calculi on a Hopf algebra  $H$  were classified by Woronowicz ([Wor89]) effectively in terms of the representations of  $D(H)$ . In [Maj98], Majid notes that braided bicovariant differential calculi on braided groups  $\underline{H}$  are classified in an entirely analogous way precisely by the appropriate double-bosonisation, which is  $T(H)$ . If one could prove a co-twisting theorem for  $T(H)$ —that as an algebra it is the tensor product of three copies of  $H$  (this is suggested by our Theorem 5.2.3 for  $T(\mathfrak{g})$ )—one would then be able to classify such braided differential calculi, for example. Such a theorem, if true, appears to be rather non-trivial to prove.

## Appendix A

# Appendix

In this appendix, we give the explicit calculations for the deletions  $(\mathfrak{g}, d, \mathfrak{g}_0, \iota)$  with  $\mathfrak{g}$  and  $\mathfrak{g}_0$  simple. In the notation of Section 3.1.1, set  $\mathfrak{b}_i = \mathfrak{g}_{[i]}$  for  $i < 0$ , the graded components of  $\mathfrak{b}$  as a graded Lie algebra. The grading gives us another way to analyse  $\mathfrak{b}$ , since we can consider the  $\mathfrak{g}_0$ -module  $\bigwedge^2 \mathfrak{b}$  and its subspaces. In particular, we can consider  $\bigwedge^2 \mathfrak{b}_{-1}$ , which will give us information about  $\mathfrak{b}_{-2}$ .

Firstly, if  $m_d = 1$ , so  $\mathfrak{b}$  is irreducible,  $\mathfrak{b}$  has zero bracket. Secondly, if  $m_d = 2$  and  $\dim \mathfrak{b}_{-2} = 1$ , there is a non-zero bracket on  $\mathfrak{b}_{-1}$  and it is a cocycle central extension of the zero bracket. For  $\dim \mathfrak{b}_{-2} = 1$  implies  $\mathfrak{b}_{-2}$  is spanned by  $\Lambda$ , the highest root in  $\mathfrak{g}$  and the grading on  $\mathfrak{b}$  tells us that if  $X_\alpha^-, X_\beta^- \in \mathfrak{b}_{-1}$  then  $[X_\alpha^-, X_\beta^-] = \delta(\alpha + \beta, \Lambda) X_\Lambda^-$  where  $\delta(\alpha + \beta, \Lambda) = 0$  if  $\alpha + \beta \neq \Lambda$  and  $\delta(\alpha + \beta, \Lambda) = c_{\alpha\beta}$  (some constant depending on  $\alpha$  and  $\beta$ ) if  $\alpha + \beta = \Lambda$ . If  $m_d \geq 2$  and  $\dim \mathfrak{b}_{-2} > 1$ , although a similar additive formula will hold, we cannot be so explicit.

The bracket  $[\cdot, \cdot]_{\mathfrak{g}} : \bigwedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$  clearly restricts to  $[\cdot, \cdot]_{\mathfrak{b}} : \bigwedge^2 \mathfrak{b} \rightarrow \mathfrak{b}$  and indeed even restricts to  $[\cdot, \cdot]_{-1} : \bigwedge^2 \mathfrak{b}_{-1} \rightarrow \mathfrak{b}_{-2}$ . Hence we can consider the kernel  $K_{-1}$  of  $[\cdot, \cdot]_{-1}$ ; which must be a sum of irreducible components of  $\bigwedge^2 \mathfrak{b}_{-1}$  (possibly zero but not all of  $\mathfrak{b}_{-1}$ ) and so we have  $\mathfrak{b}_{-2} \cong \bigwedge^2 \mathfrak{b}_{-1} / K_{-1}$ . Given the restricted number of possibilities for  $\mathfrak{b}_{-1}$  (which we know), clearly there will not be very many choices for  $\mathfrak{b}_{-2}$ , so in the case where  $\mathfrak{g}$  and  $\mathfrak{g}_0$  are simple ( $m_d \leq 3$ ) we are essentially done. In particular, if  $\mathfrak{b}_{-2} \neq \{0\}$  and  $\bigwedge^2 \mathfrak{b}_{-1}$  is irreducible, we have  $\text{Ker } [\cdot, \cdot]_{-1} \neq \bigwedge^2 \mathfrak{b}_{-1}$  so  $\text{Ker } [\cdot, \cdot]_{-1} = 0$  and  $\mathfrak{b}_{-2} \cong \bigwedge^2 \mathfrak{b}_{-1}$ .

All of the above has been classical, in the sense that it has been derived from properties of root systems.



We now consider the final structure we need on  $\mathfrak{b}$ , that of a braided-Lie bialgebra. This has been given in the proof of [Maj00, Proposition 4.5] when the quasitriangular structure on  $\mathfrak{g}$  is chosen to be the Drinfel'd-Sklyanin solution. In this case, it has the general form

$$\underline{\delta}X_{\alpha}^{-} = \sum_{\alpha=\beta+\gamma} c_{\beta\gamma} X_{\beta}^{-} \wedge X_{\gamma}^{-} \in \bigwedge^2 \mathfrak{b}.$$

By the additivity property of the multiplicity  $\text{mult}_d(-)$ , this must be zero on elements of  $\mathfrak{b}_{-1}$  since if  $\alpha = \beta + \gamma$  for some  $\beta, \gamma \in \mathfrak{b}_{-1}$  then  $\text{mult}_d(\alpha) = 2$ . However, if  $m_d \geq 2$ ,  $\underline{\delta}$  will not be zero on  $\mathfrak{b}_j$ ,  $j \leq -2$ . If  $\bigwedge^2 \mathfrak{b}_{-1}$  is irreducible, by the above,  $\mathfrak{b}_{-2} \cong \bigwedge^2 \mathfrak{b}_{-1}$  so using Schur's lemma  $\underline{\delta}$  must be an isomorphism.

We have used the above tools and the computer program LiE ([vL94]) to calculate the braided-Lie bialgebra structures arising in the deletions  $(\mathfrak{g}, d, \mathfrak{g}_0, \iota)$  for all choices of  $\mathfrak{g}$  and  $d$  such that  $\mathfrak{g}$  and  $\mathfrak{g}_0$  are simple. These calculations are given below, grouped by the value of  $m_d$  for each deletion.

For the exceptional simple Lie algebras, we have given less detail as the maps are not easily expressible in simple terms and the explicit formulæ not necessarily very informative. We wish to stress, though, that once the task of writing down the Weyl basis (or equivalently the root system) has been achieved, it is relatively simple to recover these formulæ. For a summary of the module structures, we refer the reader to Table 3.3 on page 56.

### A.1 $m_d = 1$

Recall from above that in the case  $m_d = 1$ ,  $\mathfrak{b} = \mathfrak{b}_{-1}$  is irreducible and has zero Lie algebra and braided-Lie coalgebra structures. Below we give the induced isomorphisms of  $\mathfrak{b}$  as a set of roots of  $\mathfrak{g}$  with the usual basis for  $\mathfrak{b}$  as a  $\mathfrak{g}_0$ -module of the appropriate highest weight.

Rather than numbering the cases, we will use a two-letter code corresponding to the Dynkin types of  $\mathfrak{g}$  and  $\mathfrak{g}_0$  (in that order), suppressing the rank as subscript where this is appropriate. Other notations are as in Section 3.1.1.

(AA) Deletion  $(A_{l+1}, l, A_l, \text{id})$

$\mathfrak{b}$  has highest weight  $\omega_l$  so is the natural representation of  $A_l = \mathfrak{sl}_{l+1}$  on the vector space  $V$  of dimension  $l+1$ . A basis for  $V$  is  $\{e_i \mid 1 \leq i \leq l+1\}$  and the highest weight vector is  $e_1$ . The corresponding  $\mathfrak{g}_0$ -module isomorphism is  $e_i \mapsto X_{(l-i+2)\dots(l)(l+1)}^-$ .

(BB) Deletion  $(B_{l+1}, 1, B_l, i \mapsto i+1)$

$\mathfrak{b}$  has highest weight  $\omega_1$  so is the natural representation of  $B_l = \mathfrak{so}_{2l+1}$  on the vector space  $V$  of dimension  $2l+1$ . A basis for  $V$  is given by  $\{e_i \mid 1 \leq i \leq 2l+1\}$  and the highest weight vector is  $e_1$ . The corresponding  $\mathfrak{g}_0$ -module isomorphism is

$$\begin{cases} e_i \mapsto X_{12\dots i}^- & \text{for } 1 \leq i \leq l \\ (-1)^{l+i} e_{l+i-1} \mapsto X_{12\dots(i-1)(i)(i)\dots(l+1)(l+1)}^- & \text{for } 2 \leq i \leq l+1 \\ -e_{2l+1} \mapsto X_{12\dots(l+1)}^- \end{cases}$$

(DD) Deletion  $(D_{l+1}, 1, D_l, i \mapsto i+1)$

$\mathfrak{b}$  has highest weight  $\omega_1$  so is the natural representation of  $D_l = \mathfrak{so}_{2l}$  on the vector space  $V$  of dimension  $2l$ . A basis for  $V$  is  $\{e_i \mid 1 \leq i \leq 2l\}$  and the highest weight vector is  $e_1$ . The corresponding  $\mathfrak{g}_0$ -module isomorphism is

$$\begin{cases} e_i \mapsto X_{12\dots i}^- & \text{for } 1 \leq i \leq l \\ (-1)^{l+i} e_{l+i-1} \mapsto X_{12\dots(i-1)(i)(i)\dots(l-1)(l-1)(l)(l+1)}^- & \text{for } 2 \leq i \leq l-1 \\ e_{2l-1} \mapsto X_{12\dots(l+1)}^- \\ -e_{2l} \mapsto X_{12\dots(l-1)(l+1)}^- \end{cases}$$

( $E_7 E_6$ ) Deletion  $(E_7, 7, E_6, \text{id})$

$\mathfrak{b}$  has highest weight  $\omega_6$  and is one of the dual pair of representations of  $E_6$  of dimension 27. As discussed in [Sch66] and [Bae02], these come from the action of the group  $E_6$ , as a group of determinant-preserving linear transformations, on the exceptional Jordan algebra  $\mathfrak{h}_3(\mathbb{O})$ .

(C.A) Deletion  $(C_{l+1}, l+1, A_l, i \mapsto l-i+1)$

$\mathfrak{b}$  has highest weight  $2\omega_1$  so is the symmetric square  $\text{Sym}^2(V)$  with  $V$  the  $(l+1)$ -dimensional natural representation of  $A_l$ . A basis for  $\text{Sym}^2(V)$  is given by the set  $\{e_i e_j \mid 1 \leq i \leq j \leq l+1\}$ , so the dimension of  $\text{Sym}^2(V)$  is  $\frac{1}{2}(l+1)(l+2)$ , and the highest weight vector is  $e_1^2$ . The corresponding  $\mathfrak{g}_0$ -module isomorphism is

$$\begin{cases} e_i^2 \mapsto X_{(l-i+2)(l-i+2)\dots(l)(l)(l+1)}^- & \text{for } 1 \leq i \leq l+1 \\ e_i e_j \mapsto X_{(l-j+2)(l-j+3)\dots(l-i+1)(l-i+2)(l-i+2)\dots(l)(l)(l+1)}^- & \text{for } i < j. \end{cases}$$

(D.A) Deletion  $(D_{l+1}, l+1, A_l, i \mapsto l-i+1)$

$\mathfrak{b}$  has highest weight  $\omega_2$  so is the second exterior power  $\Lambda^2(V)$  with  $V$  the  $(l+1)$ -dimensional natural representation of  $A_l$ . The dimension of  $\Lambda^2(V)$  is  $\frac{1}{2}l(l+1)$ . A basis for  $\Lambda^2(V)$  is  $\{e_i \wedge e_j \mid 1 \leq i < j \leq l+1\}$  and the highest weight vector is  $e_1 \wedge e_2$ . The corresponding  $\mathfrak{g}_0$ -module isomorphism is

$$\begin{cases} e_1 \wedge e_2 \mapsto X_{l+1}^- \\ e_1 \wedge e_j \mapsto X_{(l-j+2)\dots(l-1)(l+1)}^- & \text{for } j \geq 3 \\ e_i \wedge e_j \mapsto X_{(l-j+2)\dots(l-i+1)(l-i+2)(l-i+2)\dots(l-1)(l-1)(l)(l+1)}^- & \text{for } 2 \leq i < k \leq l+1. \end{cases}$$

( $E_6 D_5$ ) Deletion  $(E_6, 1, D_5, i \mapsto 7-i)$

$\mathfrak{b}$  has highest weight  $\omega_4$  so is the positive (half-)spin representation  $S_5^+$  of  $D_5$  (see for example [FH91, Chapter 20]). As a vector space,  $S_5^+ = \Lambda^0(V) \oplus \Lambda^2(V) \oplus \Lambda^4(V)$  with  $V$  the vector space of dimension five. Hence a basis for  $S_5^+$  is given by taking the natural bases for these pieces. The highest weight vector is  $e_1 \wedge e_2 \wedge e_3 \wedge e_4$ . The corresponding  $\mathfrak{g}_0$ -module isomorphism may easily be calculated from this.

## A.2 $m_d = 2$

The Lie algebra and braided-Lie coalgebra structures are no longer zero and we give explicit expressions for these where possible, in addition to the description following the pattern of the above.

(CC) Deletion  $(C_{l+1}, 1, C_l, i \mapsto i + 1)$

$\mathfrak{b}_{-1}$  has highest weight  $\omega_1$  so is the natural representation of  $C_l = \mathfrak{sp}_{2l}$  on the vector space  $V$  of dimension  $2l$ . A basis for  $V$  is  $\{e_i \mid 1 \leq i \leq 2l\}$  and the highest weight vector is  $e_1$ . The corresponding  $\mathfrak{g}_0$ -module isomorphism is

$$\begin{cases} e_i \mapsto X_{12\dots i}^- & \text{for } 1 \leq i \leq l \\ (-1)^{l+i-1} e_{l+i-1} \mapsto X_{12\dots(i-1)(i)(i)\dots(l)(l)(l+1)}^- & \text{for } 2 \leq i \leq l \\ e_{2l} \mapsto X_{12\dots(l+1)}^- \end{cases}$$

$\mathfrak{b}_{-2}$  has highest weight  $\omega_0 = [0, 0, \dots, 0]$  so is the trivial representation. We can see this by a dimension calculation. So, as described above,  $\mathfrak{b}_{-2}$  is spanned by the highest root,  $X_{1122\dots(l)(l)(l+1)}^- = \varsigma$ .

The bracket on  $\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_{-2}$  is a cocycle central extension of the zero bracket on  $\mathfrak{b}_{-1}$ , with  $[e_i, (-e_{l+i})] = c_i \varsigma$  for  $1 \leq i \leq l-1$  and  $[e_l, e_{2l}] = c_l \varsigma$ , where the  $c_i$ ,  $1 \leq i \leq l$ , are constants. The braided-Lie cobracket is zero on elements of  $\mathfrak{b}_{-1}$ , as discussed previously, and

$$\underline{\delta}\varsigma = \sum_{i=1}^l \gamma_i (e_i \wedge e_{l+i})$$

for some constants  $\gamma_i$ .

We have  $\bigwedge^2 \mathfrak{b}_{-1} \cong V(\omega_2) \oplus V(\omega_0)$  ( $V(\omega)$  is the representation of  $C_l$  with highest weight  $\omega$ ) and we see that we have  $\text{Ker} [\ , \ ] \cong V(\omega_2)$ ,  $\mathfrak{b}_{-2} \cong V(\omega_0) = \mathbb{C}$ .

( $E_8 E_7$ ) Deletion  $(E_8, 8, E_7, \text{id})$

$\mathfrak{b}_{-1}$  has highest weight  $\omega_7$  and is the smallest non-trivial representation of  $E_7$ . This may be realised by a Freudenthal triple system (see [Bae02] and the references therein). The dimension of  $\mathfrak{b}_{-1}$  is 56.

$\mathfrak{b}_{-2}$  has highest weight  $\omega_0$  so is the trivial representation, by a dimension calculation. It is spanned by the highest root in  $E_8$ ,  $X_{(2,3,4,6,5,4,3,2)}^-$ .

The bracket on  $\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_{-2}$  is again a cocycle central extension of the zero bracket on  $\mathfrak{b}_{-1}$  and has the additive form described previously. Similarly, the braided-Lie cobracket is non-zero only on  $\mathfrak{b}_{-2}$  and has the additive form.

*Note:* One might consider that this deletion provides the most natural basis for the 56-dimensional representation of  $E_7$ .

$(E_6 A_5)$  Deletion  $(E_6, 2, A_5, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 6 \end{pmatrix})$

$\mathfrak{b}_{-1}$  has highest weight  $\omega_3$  so is the third exterior power  $\bigwedge^3(V)$  with  $V$  the six-dimensional natural representation of  $A_5$ . The dimension of  $\bigwedge^3(V)$  is 20. A basis for  $\bigwedge^3(V)$  is  $\{e_i \wedge e_j \wedge e_k \mid 1 \leq i < j < k \leq 6\}$  and the highest weight vector is  $e_1 \wedge e_2 \wedge e_3$ . The corresponding  $\mathfrak{g}_0$ -module isomorphism may be calculated from this.

$\mathfrak{b}_{-2}$  has highest weight  $\omega_0$ , so is the trivial representation, by a dimension calculation. It is spanned by the highest root in  $E_6$ ,  $X_{(1,2,2,3,2,1)}^-$ . However, as we will see, we should consider  $\mathfrak{b}_{-2}$  to be  $\bigwedge^6(V)$  with  $V$  as before, spanned by  $e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6$ .

The bracket on  $\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_{-2}$  is given by the map

$$\wedge : \mathfrak{b}_{-1} \otimes \mathfrak{b}_{-1} \rightarrow \mathfrak{b}_{-2}, (e_{i_1} \wedge e_{j_1} \wedge e_{k_1}) \otimes (e_{i_2} \wedge e_{j_2} \wedge e_{k_2}) \mapsto e_{i_1} \wedge e_{j_1} \wedge e_{k_1} \wedge e_{i_2} \wedge e_{j_2} \wedge e_{k_2},$$

that is, the wedge product. The bracket is zero on all other elements of  $\mathfrak{b}^{\otimes 2}$ . The braided-Lie cobracket is a map

$$\underline{\delta} : \mathfrak{b}_{-2} \rightarrow \mathfrak{b}_{-1} \wedge \mathfrak{b}_{-1} \cong \bigwedge^3(V) \wedge \bigwedge^3(V) \cong \bigwedge^6(V) \cong \mathfrak{b}_{-2}$$

so must be a non-zero scalar multiple of the identity.

$(F_4 C_3)$  Deletion  $(F_4, 1, C_3, i \mapsto 5 - i)$

$\mathfrak{b}_{-1}$  has highest weight  $\omega_3$  and is described as the kernel of the contraction map  $\varphi_3 : \bigwedge^3(V) \rightarrow V$  for  $V$  the six-dimensional natural representation of  $C_3 = \mathfrak{sp}_3$  (see, for example, [FH91, p. 258]). The dimension of  $\mathfrak{b}_{-1}$  is 14.

$\mathfrak{b}_{-2}$  has highest weight  $\omega_0$  so is the trivial representation, by a dimension calculation. It is spanned by the highest root in  $F_4$ ,  $X_{(2,3,4,2)}^- = \varsigma$ .

The bracket on  $\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_{-2}$  is again a cocycle central extension of the zero bracket on  $\mathfrak{b}_{-1}$  and has the additive form described previously. Similarly, the braided-Lie cobracket is non-zero only on  $\varsigma$  and has the additive form.

$(G_2A_1)$ (a) Deletion  $(G_2, 2, A_1, \text{id})$

$\mathfrak{b}_{-1}$  has highest weight  $3\omega_1$  so is the third symmetric power  $\text{Sym}^3(V)$  with  $V$  the two-dimensional natural representation of  $A_1$ . A basis for  $\text{Sym}^3(V)$  is given by the set  $\{e_1^3, e_1^2e_2, e_1e_2^2, e_2^3\}$  and the highest weight vector is  $e_1^3$ . The dimension of  $\text{Sym}^3(V)$  is four. The corresponding  $\mathfrak{g}_0$ -module isomorphism is

$$\begin{aligned} e_1^3 &\mapsto X_2^-, \\ e_1^2e_2 &\mapsto X_{12}^-, \\ e_1e_2^2 &\mapsto X_{112}^-, \\ e_2^3 &\mapsto X_{1112}^-. \end{aligned}$$

$\mathfrak{b}_{-2}$  has highest weight  $\omega_0$  so is the trivial representation, by a dimension calculation. It is spanned by the highest root in  $G_2$ ,  $X_{1112}^-$ . We can consider  $\mathfrak{b}_{-2}$  to be spanned by  $e_1^3e_2^3$ , for the following reason.

The bracket on  $\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_{-2}$  is a cocycle central extension of the zero bracket on  $\mathfrak{b}_{-1}$ , given explicitly by  $[e_1^ie_2^j, e_1^ke_2^l] = \delta_{(i+k),3} \delta_{(j+l),3} e_1^3e_2^3$ . The braided-Lie cobracket is

$$\underline{\delta}(e_1^3e_2^3) = \sum_{\substack{i,j,k,l=0 \\ i+k=3 \\ j+l=3}}^3 \gamma_{ijkl} e_1^ie_2^j \wedge e_1^ke_2^l$$

for some non-zero constants  $\gamma_{ijkl}$ .

*Note:* This case has been covered as Example 4.6 in [Maj00].

$(E_7D_6)$  Deletion  $(E_7, 1, D_6, i \mapsto 8 - i)$

$\mathfrak{b}_{-1}$  has highest weight  $\omega_5$  so is the negative (half-)spin representation  $S_6^-$  of  $D_6$  (see for example [FH91, Chapter 20]). The dimension of  $S_6^-$  is 32. A basis for

$S_6^- = V \oplus \bigwedge^3(V) \oplus \bigwedge^5(V)$  (as vector spaces;  $V$  the vector space of dimension six) is given by taking the natural bases for these pieces and the highest weight vector is  $e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5$ .

$\mathfrak{b}_{-2}$  has highest weight  $\omega_0$  so is the trivial representation. In what follows, we see that  $\bigwedge^6(V)$ , spanned by  $e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6$ , is the correct choice of basis for  $\mathfrak{b}_{-2}$ .

The bracket on  $\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_{-2}$  is given by the wedge product, i.e. is non-zero on the subspaces  $V \wedge \bigwedge^5(V)$  and  $\bigwedge^3(V) \wedge \bigwedge^3(V)$  of  $\mathfrak{b}_{-1} \wedge \mathfrak{b}_{-1}$ . The braided-Lie cobracket will be a non-zero map  $\underline{\delta} : \bigwedge^6(V) \rightarrow \bigwedge^6(V)$ , i.e. is a non-zero scalar multiple of the identity.

(BA) Deletion  $(B_{l+1}, l+1, A_l, i \mapsto l-i+1)$

$\mathfrak{b}_{-1}$  has highest weight  $\omega_1$  so is the natural representation of  $A_l$  on the vector space  $V$  of dimension  $l+1$ . A basis for  $V$  is  $\{e_i \mid 1 \leq i \leq l+1\}$  and the highest weight vector is  $e_1$ . The corresponding  $\mathfrak{g}_0$ -module isomorphism is  $e_i \mapsto X_{i(i+1)\dots(l+1)}^-$ , for  $1 \leq i \leq l+1$ .

$\mathfrak{b}_{-2}$  has highest weight  $\omega_2$  so is the second exterior power  $\bigwedge^2(V)$  with  $V$  as before. The dimension of  $\bigwedge^2(V)$  is  $\frac{1}{2}l(l+1)$ . A basis for  $\bigwedge^2(V)$  is  $\{e_i \wedge e_j \mid 1 \leq i < j \leq l+1\}$  and the highest weight vector is  $e_1 \wedge e_2$ . We may deduce this from the following.

The bracket on  $\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_{-2}$  is non-zero: for example, there exists  $X_\alpha^-$  such that  $[X_\alpha^-, X_{l+1}^-] \in \mathfrak{g}^\Lambda$  where  $\Lambda$  is the highest root in  $\mathfrak{g} = B_{l+1}$ . Clearly, we have the isomorphism  $\mathfrak{b}_{-2} \cong \bigwedge^2 \mathfrak{b}_{-1} / \text{Ker} [\ , \ ]_{-1}$  but  $\bigwedge^2 \mathfrak{b}_{-1} = \bigwedge^2(V)$  is irreducible. Since  $\text{Ker} [\ , \ ]_{-1} \neq \bigwedge^2 \mathfrak{b}_{-1}$ , we see that  $\mathfrak{b}_{-2} \cong \bigwedge^2 \mathfrak{b}_{-1} = \bigwedge^2(V)$ . Further, the bracket is  $[\ , \ ]_{-1} = \wedge : V \otimes V \rightarrow \bigwedge^2(V)$ . The braided-Lie cobracket  $\underline{\delta} : \mathfrak{b}_{-2} \rightarrow \bigwedge^2 \mathfrak{b}_{-1}$  is an isomorphism.

The  $\mathfrak{g}_0$ -module isomorphism is given on  $\mathfrak{b}_{-2}$  by

$$e_i \wedge e_j \mapsto X_{i(i+1)\dots(j-1)(j)(j+1)\dots(l+1)(l+1)}^-$$

for  $1 \leq i < j \leq l+1$ .

$(E_7 A_6)$  Deletion  $(E_7, 2, A_6, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 5 & 6 & 7 \end{pmatrix})$

$\mathfrak{b}_{-1}$  has highest weight  $\omega_3$  so is the third exterior power  $\bigwedge^3(V)$  with  $V$  the seven-dimensional natural representation of  $A_6$ . The dimension of  $\bigwedge^3(V)$  is 35. A basis for  $V$  is  $\{e_i \wedge e_j e_k \mid 1 \leq i < j < k \leq 7\}$  and the highest weight vector is  $e_1 \wedge e_2 \wedge e_3$ .

$\mathfrak{b}_{-2}$  has highest weight  $\omega_6$  by considering the module decomposition

$$\bigwedge^2 \mathfrak{b}_{-1} = V([0, 1, 0, 1, 0, 0]) \oplus \bigwedge^6(V)$$

(we use a formula in [FH91, Chapter 15]) and a dimension calculation. We use the usual natural basis for  $\bigwedge^6(V)$  rather than a basis in terms of the dual of  $V$ , even though  $\bigwedge^6(V) \cong V^*$ . The dimension of  $\bigwedge^6(V)$  is seven.

The bracket on  $\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_{-2}$  is given by the wedge product map

$$\wedge : \mathfrak{b}_{-1} \otimes \mathfrak{b}_{-1} = \bigwedge^3(V) \otimes \bigwedge^3(V) \rightarrow \mathfrak{b}_{-2} = \bigwedge^6(V).$$

The kernel of  $\wedge$  is  $V([0, 1, 0, 1, 0, 0])$ . The braided-Lie cobracket  $\underline{\delta}$  is an isomorphism.

$(F_4 B_3)$  Deletion  $(F_4, 4, B_3, \text{id})$

$\mathfrak{b}_{-1}$  has highest weight  $\omega_3$  so is the eight-dimensional spinor representation  $S_3$  of  $B_3 = \mathfrak{so}_7$ . A basis for  $S_3 = \bigoplus_{i=0}^3 \bigwedge^i(V)$  (as vector spaces;  $V$  the vector space of dimension three) is given by taking the natural basis for each piece and the highest weight vector is  $e_1 \wedge e_2 \wedge e_3$ .

$\mathfrak{b}_{-2}$  has highest weight  $\omega_1$ , by considering the module decomposition

$$\bigwedge^2 \mathfrak{b}_{-1} = \bigwedge^2(S_3) \cong W \oplus \bigwedge^2(W)$$

for  $W$  the seven-dimensional natural representation of  $B_3$  and a dimension calculation. We obtain this decomposition by examining the above description of  $S_3$ . So,  $\mathfrak{b}_{-2}$  is isomorphic to the natural representation,  $W$ .

The bracket on  $\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_{-2}$  does not seem to have an interpretation as a natural map on  $\bigwedge^2(S_3)$ .



$(E_8 D_7)$  Deletion  $(E_8, 1, D_7, i \mapsto 9 - i)$

$\mathfrak{b}_{-1}$  has highest weight  $\omega_6$  so is the positive (half-)spin representation  $S_7^+$  of  $D_7$ . The dimension of  $S_7^+$  is 64. As a vector space, we have

$$S_7^+ = \bigwedge^{\text{even}}(V) = \bigoplus_{i=0,2,4,6} \bigwedge^i(V)$$

with  $V$  the vector space of dimension seven so a basis is given by taking the natural basis for each piece. The highest weight vector is  $e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6$ .

$\mathfrak{b}_{-2}$  has highest weight  $\omega_1$ , so is the 14-dimensional natural representation  $W$  of  $D_7$ . A basis for  $W$  is  $\{e_i \mid 1 \leq i \leq 14\}$  and the highest weight vector is  $e_1$ . We obtain this from the decomposition  $\bigwedge^2 \mathfrak{b}_{-1} = \bigwedge^2(S_7^+) \cong \bigwedge^5(W) \oplus W$  and a dimension calculation.

### A.3 $m_d = 3$

$(E_8 A_7)$  Deletion  $(E_8, 2, A_7, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix})$

$\mathfrak{b}_{-1}$  has highest weight  $\omega_3$  so is the third exterior power  $\bigwedge^3(V)$  with  $V$  the eight-dimensional natural representation of  $A_7$ . The dimension of  $\bigwedge^3(V)$  is 56. We take the natural basis for  $\bigwedge^3(V)$  and the highest weight vector is  $e_1 \wedge e_2 \wedge e_3$ .

$\mathfrak{b}_{-2}$  has highest weight  $\omega_6$  so is the sixth exterior power  $\bigwedge^6(V)$ , with  $V$  as before. The dimension of  $\bigwedge^6(V)$  is 28. We take the natural basis and the highest weight vector is  $e_1 \wedge \cdots \wedge e_6$ . We obtain this from the decomposition

$$\bigwedge^2 \mathfrak{b}_{-1} = \bigwedge^2(\bigwedge^3(V)) \cong V([0, 1, 0, 1, 0, 0, 0]) \oplus \bigwedge^6(V)$$

and calculating dimensions.

$\mathfrak{b}_{-3}$  has highest weight  $\omega_1$ , so is the eight-dimensional natural representation  $V$ . The highest weight vector is  $e_1$ . We see this since the tensor product of  $\mathfrak{b}_{-1} \otimes \mathfrak{b}_{-2}$  decomposes as

$$\bigwedge^3(V) \otimes \bigwedge^6(V) \cong V([0, 0, 1, 0, 0, 1, 0]) \oplus V([0, 1, 0, 0, 0, 0, 1]) \oplus V$$

so, by the same arguments about the kernel of the bracket map, we can use a dimension calculation as before.

$(G_2, A_1)(b)$  Deletion  $(G_2, 1, A_1, \left(\frac{1}{2}\right))$

$\mathfrak{b}_{-1}$  has highest weight  $\omega_1$  so is the two-dimensional natural representation of  $A_1 = \mathfrak{sl}_2$ . A basis for  $V$  is  $\{e_1, e_2\}$  and the highest weight vector is  $e_1$ . We have  $e_1 \mapsto X_1^-$ ,  $e_2 \mapsto X_{12}^-$ .

$\mathfrak{b}_{-2}$  has highest weight  $\omega_0$  so is the trivial representation, spanned by  $\varsigma$ , say. This is since  $\bigwedge^2(V) \cong \mathbb{C}$  and  $\text{Ker}[\ , \ ]_{-1} = 0$  (the bracket is non-zero on  $\mathfrak{b}_{-1}$ ). We have  $\varsigma \mapsto X_{112}^-$ .

$\mathfrak{b}_{-3}$  has highest weight  $\omega_1$  so is another copy of the natural representation  $V$ , with basis  $\{f_1, f_2\}$ . The highest weight vector is  $f_1$ . This is obtained from a direct examination of the root system of  $G_2$ , giving  $f_1 \mapsto X_{1112}^-$  and  $f_2 \mapsto X_{11122}^-$ .

The bracket in these bases is

$$\begin{aligned} [e_1, e_2] &= c_1 \varsigma, \\ [e_1, \varsigma] &= c_2 f_1, \\ [e_2, \varsigma] &= c_3 f_2 \end{aligned}$$

for some constants  $c_i$ .

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